Plasma notes: Ohm's Law and Hall MHD

by E. Alec Johnson, October, 2007.

1 Boltzmann equation

The Boltzmann equation for species s is an evolution equation for particle density in phase space. Phase space is a collection of coordinates which bijectively specify the state of a particle. For point particles, such as electrons and protons, points in phase space are represented by $(\mathbf{x}, \widetilde{\mathbf{v}})$, the position and (proper) velocity of the particle. (Ignore the blue text and wide tildes if you do not care about the special relativistic regime. If you do care about relativistic effects the blue text is incomplete and needs to be fixed; read Cercignani's 2002 book on the Relativistic Boltzmann Equation instead. So just ignore the blue text.) For molecules with other degrees of freedom, phase space must also include other variables which specify not only translational modes, but such state quantities as rotational and vibrational modes. The rate of motion of a particle through phase space is $(\dot{\mathbf{x}}, \dot{\widetilde{\mathbf{v}}}) =: (\mathbf{v}, \mathbf{a})$. The Boltzmann equation specifies particle balance in phase space:

$$\partial_t f_s + \nabla \cdot (\mathbf{v} f_s) + \nabla_{\widetilde{\mathbf{v}}} \cdot (\mathbf{a}_s f_s) = C_s.$$

Here s is the species index, $\mathbf{x}$ is position in space, $\mathbf{v}$ is particle velocity, $\widetilde{\mathbf{v}} = \gamma \mathbf{v}$ is proper velocity, $f_s = f_s^{\mathrm{p}}(t, \mathbf{x}, \widetilde{\mathbf{v}})$ is the particle density of species s (i.e., $f_s^{\mathrm{p}}(t, \mathbf{x}, \widetilde{\mathbf{v}})\, \mathrm{d}\mathbf{x}\, \mathrm{d}\widetilde{\mathbf{v}}$ is the number of particles in the infinitesimal box $\mathrm{d}\mathbf{x}\, \mathrm{d}\widetilde{\mathbf{v}}$), $\mathbf{a}_s = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{g} = d_t \mathbf{v}_s$ is the rate of change of the proper velocity of a particle (where q_s is charge per particle, m_s is mass per particle, $\mathbf{E}$ is electric field, $\mathbf{B}$ is magnetic field, $\mathbf{g}$ is gravitational field), and $C_s = C_s^{\mathrm{p}}$, known as the *collision operator*, is the rate of change in particle density due purely to collisions (within species s or with other species). Note that C_s is an operator which depends on the distribution functions of each species regarded as a function of velocity (but not of position or time): $C_s[(\mathbf{v} \mapsto f_\beta(t, \mathbf{x}, \mathbf{v}))_{\beta \in \Sigma}]$, where Σ is the set of all species indices.

1.1 Notes on the relatistic Boltzmann equation

We use $d\tau$ to denote the elapse of proper time. By the invariance of the Lorentz metric under inertial transformations, $d\tau^2 = dt^2 - (d\mathbf{x}/c)^2$. Dividing alternatively by $d\tau^2$ and dt^2 shows that the Lorentz factor $\gamma := \frac{dt}{d\tau}$ as a function of velocity and as a function of proper velocity is $\gamma(\mathbf{v}) = (1 - (\frac{\mathbf{v}}{c})^2)^{-1/2} = \widetilde{\gamma}(\widetilde{\mathbf{v}}) = (1 + (\frac{\widetilde{\mathbf{v}}}{c})^2)^{1/2}$.

1.1.1 Differentation of the Lorentz factor

To express derivatives of the Lorentz factor in terms of derivatives of the proper velocity, we differentiate the relation $\gamma^2 = 1 + \widetilde{\mathbf{v}}^2$. We get $\gamma\, d\gamma = (d\widetilde{\mathbf{v}}) \cdot \widetilde{\mathbf{v}}$, i.e.,

$$d\widetilde{\gamma}(\widetilde{\mathbf{v}}) = (d\widetilde{\mathbf{v}}) \cdot \frac{\widetilde{\mathbf{v}}}{\widetilde{\gamma}(\widetilde{\mathbf{v}})}.$$

1.1.2 Derivation of the Boltzmann equation

$$
\begin{aligned}
&\partial_t f_s + \nabla_{\mathbf{x}} \cdot \big((d_t\mathbf{x}) f_s\big) + \nabla_{\widetilde{\mathbf{v}}} \cdot \big(d_t \widetilde{\mathbf{v}}_s f_s\big) = C_s.\\
&\partial_t f_s + \nabla_{\mathbf{x}} \cdot \big(\frac{(d_\tau \mathbf{x})}{\gamma_s} f_s\big) + \nabla_{\widetilde{\mathbf{v}}} \cdot \big(\frac{d_\tau \widetilde{\mathbf{v}}_s}{\gamma_s} f_s\big) = C_s.\\
&\partial_t f_s + \nabla \cdot (\mathbf{v} f_s) + \nabla_{\widetilde{\mathbf{v}}} \cdot (\mathbf{a}_s f_s) = C_s.
\end{aligned}
$$

1.1.3 Conventions of interpretation

Note that by multiplying by m_s or q_s and, respectively, by making the redefinitions $f_s := f_s^{\mathrm{m}} := m_s f_s^{\mathrm{p}}$ and $C_s := C_s^{\mathrm{m}} := m_s C_s^{\mathrm{p}}$, or $f_s := f_s^{\mathrm{q}} := q_s f_s^{\mathrm{p}}$ and $C_s := C_s^{\mathrm{q}} := q_s C_s^{\mathrm{p}}$, we can also regard the Boltzmann equation as a statement of conservation of mass or charge.

Henceforth, we drop the default species index s (except as a reminder, particularly when making definitions) until we consider multiple species. So there is an implicit species index s on most variables (except for the independent variables $t, \mathbf{x}, \widetilde{\mathbf{v}}$ (and $\chi(\widetilde{\mathbf{v}})$ below) and the field variables $\mathbf{E}, \mathbf{B}, \mathbf{g}$).

2 Species balance laws

Henceforth we view the Boltzmann equation by default as mass-conservation in phase space. Taking moments of the Boltzmann equation yields balance laws for density, momentum, and energy. Define $\int_{\widetilde{\mathbf{v}}} := \int_{\widetilde{\mathbf{v}} \in \mathbb{R}^3}$. Define $\rho_s := \int_{\widetilde{\mathbf{v}}} f_s$, $\langle M \rangle_s := \frac{\int_{\widetilde{\mathbf{v}}} M f_s}{\rho_s}$, the average velocity $\mathbf{u}_s = \langle \mathbf{v} \rangle_s$, and the thermal velocity $\mathbf{c}_s := \mathbf{v}_s - \mathbf{u}_s$. Define the average proper velocity $\widetilde{\mathbf{u}}_s = \langle \widetilde{\mathbf{v}} \rangle_s$ and the thermal proper velocity $\widetilde{\mathbf{c}}_s := \widetilde{\mathbf{v}}_s - \widetilde{\mathbf{u}}_s$. Let χ be a moment function (power) of $\mathbf{v}$ (e.g., 1, $\mathbf{v}$, v^2, or $\mathbf{vv}$). To compute moments we multiply the Boltzmann equation by χ and integrate over $\widetilde{\mathbf{v}}$:

$$
\begin{array}{lllll}
\chi = \chi: & \int_{\widetilde{\mathbf{v}}} \chi \partial_t f & + \int_{\widetilde{\mathbf{v}}} \chi \nabla_{\mathbf{x}} \cdot (\mathbf{v} f) & + \int_{\widetilde{\mathbf{v}}} \chi \nabla_{\widetilde{\mathbf{v}}} \cdot (\mathbf{a} f) & = \int_{\widetilde{\mathbf{v}}} \chi C_s \\
 & \partial_t \int_{\widetilde{\mathbf{v}}} \chi f & + \nabla_{\mathbf{x}} \cdot \int_{\widetilde{\mathbf{v}}} \mathbf{v} \chi f & + \cancel{\int_{\widetilde{\mathbf{v}}} \nabla_{\widetilde{\mathbf{v}}} \cdot (\mathbf{a} \chi f)} - \int_{\widetilde{\mathbf{v}}} f \mathbf{a} \cdot \nabla_{\widetilde{\mathbf{v}}} \chi & = \int_{\widetilde{\mathbf{v}}} \chi C_s \\
 & \partial_t (\rho \langle \chi \rangle) & + \nabla_{\mathbf{x}} \cdot (\rho \langle \mathbf{v} \chi \rangle) & + -\rho \langle \mathbf{a} \cdot \nabla_{\widetilde{\mathbf{v}}} \chi \rangle & = \int_{\widetilde{\mathbf{v}}} \chi C_s
\end{array}
$$

Defining $\nabla := \nabla_{\mathbf{x}}$, we have the moment equations:

$$
\begin{array}{lllll}
\chi = \chi: & \partial_t (\rho \langle \chi \rangle) & + \nabla \cdot (\rho \langle \mathbf{v} \chi \rangle) & = \rho \langle \mathbf{a} \cdot \nabla_{\widetilde{\mathbf{v}}} \chi \rangle & + \int_{\widetilde{\mathbf{v}}} C_s \chi, \\
\chi = 1: & \partial_t \rho & + \nabla \cdot (\rho \mathbf{u}) & = 0 & + \int_{\widetilde{\mathbf{v}}} C_s =: S_s, \\
\chi = \widetilde{\mathbf{v}}: & \partial_t (\rho \widetilde{\mathbf{u}}) & + \nabla \cdot (\rho \langle \mathbf{v} \widetilde{\mathbf{v}} \rangle) & = \rho \langle \mathbf{a} \rangle & + \int_{\widetilde{\mathbf{v}}} C_s \widetilde{\mathbf{v}} =: \mathbf{A}_s, \\
\chi = c\widetilde{\gamma}(\widetilde{\mathbf{v}}) = \widetilde{\mathbf{v}}^0: & \partial_t (\rho \langle c\gamma \rangle) & + \nabla \cdot (\rho \langle \mathbf{v} c\gamma \rangle) & = \rho \langle \mathbf{a} \cdot \mathbf{v}/c \rangle & + \int_{\widetilde{\mathbf{v}}} C_s c\gamma, \\
\chi = \widetilde{\mathbf{v}}^2/2: & \partial_t (\rho \langle \widetilde{\mathbf{v}}^2 \rangle/2) & + \nabla \cdot (\rho \langle \mathbf{v} \widetilde{\mathbf{v}}^2 \rangle/2) & = \rho \langle \mathbf{a} \cdot \widetilde{\mathbf{v}} \rangle & + \int_{\widetilde{\mathbf{v}}} C_s \widetilde{\mathbf{v}}^2/2 =: Q_s, \\
\chi = \widetilde{\mathbf{v}}\widetilde{\mathbf{v}}: & \partial_t (\rho \langle \widetilde{\mathbf{v}}\widetilde{\mathbf{v}} \rangle) & + \nabla \cdot (\rho \langle \mathbf{v} \widetilde{\mathbf{v}}\widetilde{\mathbf{v}} \rangle) & = \rho \langle \mathbf{a} \widetilde{\mathbf{v}} + \widetilde{\mathbf{v}} \mathbf{a} \rangle & + \int_{\widetilde{\mathbf{v}}} C_s \widetilde{\mathbf{v}}\widetilde{\mathbf{v}}.
\end{array}
$$

Here c is the speed of light, S_s is the rate of production of mass of species s due to collisions, and $\mathbf{A}_s = \widetilde{\mathbf{u}} S_s + (\mathbf{R}_s := \int_{\widetilde{\mathbf{c}}} \widetilde{\mathbf{c}} C_s)$ is the rate of production of momentum due to collisions, where $\mathbf{R}_s$ is the resistive force on species s due to collisions. $Q_s = S_s \widetilde{\mathbf{u}}^2/2 + \mathbf{R}_s \cdot \widetilde{\mathbf{u}} + (\mathcal{H}_s := \int_{\widetilde{\mathbf{c}}} \widetilde{\mathbf{c}}^2 C_s/2)$ is the rate of energy production due to collisions with other species, where $\mathbf{R}_s \cdot \widetilde{\mathbf{u}}$ is the rate of work due to the resistive force and $\mathcal{H}_s$ is the rate of production of heat in species s due to collisions.

2.1 $\chi = 1$: Balance of particles, mass, and charge

In the moment equations we can regard ρ as the particle number density $n_s := \int_{\widetilde{\mathbf{v}}} f_s^{\mathrm{p}}$, the mass density $\rho_s := \int_{\widetilde{\mathbf{v}}} f_s^{\mathrm{m}}$, or the charge density $\sigma_s := \int_{\widetilde{\mathbf{v}}} f_s^{\mathrm{q}}$. Define the current $\mathbf{J}_s := \sigma_s \mathbf{u}_s$. Define $\delta_t^s := a \mapsto \partial_t a + \nabla\cdot(\mathbf{u}_s a)$, the *transport* (or *bulk*) *derivative for species s*. Then we can write particle, mass, and charge balance as

$$\begin{aligned}
\delta_t^s(n) &= \partial_t n + \nabla\cdot(n\mathbf{u}) = S_s^{\mathrm{p}} := \int_v C_s^{\mathrm{p}},\\
\delta_t^s(\rho) &= \partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = S_s := S_s^{\mathrm{m}} := \int_v C_s^{\mathrm{m}} = m S_s^{\mathrm{p}},\\
\delta_t^s(\sigma) &= \partial_t \sigma + \nabla\cdot\mathbf{J} \qquad = S_s^{\mathrm{q}} := q S_s^{\mathrm{p}}.
\end{aligned}$$

2.2 $\chi = \widetilde{\mathbf{v}}$: Balance of momentum (i.e. mass flux) and current (i.e. charge flux)

Recall that $\mathbf{c}_s := \mathbf{v}_s - \mathbf{u}_s$ is the *thermal velocity* of species s. Observe that $\langle\mathbf{c}\rangle = \langle\mathbf{v}-\mathbf{u}\rangle = \langle\mathbf{v}\rangle - \mathbf{u} = 0$. Likewise, $\widetilde{\mathbf{c}}_s := \widetilde{\mathbf{v}}_s - \widetilde{\mathbf{u}}_s$ is the *proper thermal velocity* of species s, and observe that $\langle\widetilde{\mathbf{c}}\rangle = \langle\widetilde{\mathbf{v}} - \widetilde{\mathbf{u}}\rangle = \langle\widetilde{\mathbf{v}}\rangle - \widetilde{\mathbf{u}} = 0$. Use $\langle\mathbf{v}\widetilde{\mathbf{v}}\rangle = \langle(\mathbf{u}+\mathbf{c})(\widetilde{\mathbf{u}}+\widetilde{\mathbf{c}})\rangle = \mathbf{u}\widetilde{\mathbf{u}} + \langle\mathbf{c}\widetilde{\mathbf{c}}\rangle$. Use $\langle\mathbf{a}\rangle = \frac{q}{m}(\mathbf{E}+\mathbf{u}\times\mathbf{B}) + \mathbf{g}$. Define the pressure tensor $\mathbb{P}_s := \rho_s\langle\mathbf{c}_s\widetilde{\mathbf{c}}_s\rangle$, i.e. the flux of (thermal) momentum across a boundary convected with the species velocity $\mathbf{u}_s$, and define the electrokinetic pressure tensor $\mathbb{P}_s^{\mathrm{q}} := \sigma_s\langle\mathbf{c}_s\widetilde{\mathbf{c}}_s\rangle$, i.e. the flux of (thermal) charge flux across a boundary convected with the species velocity $\mathbf{u}_s$. The first moment equation becomes

$$\begin{aligned}
&\partial_t(\rho\widetilde{\mathbf{u}}) \quad + \nabla\cdot(\rho\mathbf{u}\widetilde{\mathbf{u}} + \underbrace{\rho\langle\mathbf{c}\widetilde{\mathbf{c}}\rangle}_{\mathbb{P}}) = \underbrace{nq(\mathbf{E}+\mathbf{u}\times\mathbf{B})}_{\sigma\mathbf{E}+\mathbf{J}\times\mathbf{B}} \quad + \rho\mathbf{g} + \mathbf{A}_s,\ \text{i.e.,}\\
&\partial_t(\underbrace{\sigma\widetilde{\mathbf{u}}}_{\widetilde{\mathbf{J}}}) + \nabla\cdot(\sigma\mathbf{u}\widetilde{\mathbf{u}} + \underbrace{\sigma\langle\mathbf{c}\widetilde{\mathbf{c}}\rangle}_{\mathbb{P}^{\mathrm{q}}}) = \underbrace{n\frac{q^2}{m}(\mathbf{E}+\mathbf{u}\times\mathbf{B})}_{\frac{q}{m}(\sigma\mathbf{E}+\mathbf{J}\times\mathbf{B})} + \sigma\mathbf{g} + (\mathbf{A}_s^{\mathrm{q}} := (q/m)\mathbf{A}_s).
\end{aligned}$$

2.3 $\chi = \widetilde{\mathbf{v}}^2/2$: Classical balance of energy

Use $\langle\mathbf{v}\widetilde{\mathbf{v}}^2\rangle = \langle(\mathbf{u}+\mathbf{c})(\widetilde{\mathbf{u}}+\widetilde{\mathbf{c}})\cdot(\widetilde{\mathbf{u}}+\widetilde{\mathbf{c}})\rangle = \mathbf{u}\widetilde{\mathbf{u}}^2 + \mathbf{u}\langle\widetilde{\mathbf{c}}^2\rangle + 2\widetilde{\mathbf{u}}\cdot\langle\mathbf{c}\widetilde{\mathbf{c}}\rangle + \langle\mathbf{c}\widetilde{\mathbf{c}}^2.\rangle$ Use $\langle\widetilde{\mathbf{v}}^2\rangle = \widetilde{\mathbf{u}}^2 + \langle\widetilde{\mathbf{c}}^2\rangle$. Use $\langle\mathbf{a}\cdot\widetilde{\mathbf{v}}\rangle = \langle(\frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})+\mathbf{g})\cdot\widetilde{\mathbf{v}}\rangle = \langle\frac{q}{m}\widetilde{\mathbf{v}}\cdot\mathbf{E} + \widetilde{\mathbf{v}}\cdot\mathbf{g}\rangle = \frac{q}{m}\widetilde{\mathbf{u}}\cdot\mathbf{E} + \widetilde{\mathbf{u}}\cdot\mathbf{g} = \langle\mathbf{a}\rangle\cdot\widetilde{\mathbf{v}}$. The second moment equation becomes

$$\partial_t\Big(\underbrace{\underbrace{\rho\frac{\widetilde{\mathbf{u}}^2}{2}}_{\mathcal{E}^{\mathrm{k}}} + \underbrace{\rho\frac{\langle\widetilde{\mathbf{c}}^2\rangle}{2}}_{\mathcal{E}^{\mathrm{t}}}}_{\mathcal{E}}\Big) + \nabla\cdot\Big(\mathbf{u}\underbrace{\rho\big(\frac{\widetilde{\mathbf{u}}^2}{2} + \frac{\langle\widetilde{\mathbf{c}}^2\rangle}{2}\big)}_{\mathcal{E}} + \widetilde{\mathbf{u}}\cdot\underbrace{\rho\langle\mathbf{c}\widetilde{\mathbf{c}}\rangle}_{\mathbb{P}} + \underbrace{\rho\langle\mathbf{c}\frac{\widetilde{\mathbf{c}}^2}{2}\rangle}_{\mathbf{q}}\Big) = \underbrace{nq\widetilde{\mathbf{u}}}_{\widetilde{\mathbf{J}}}\cdot\mathbf{E} + \underbrace{nm}_{\rho}\widetilde{\mathbf{u}}\cdot\mathbf{g} + Q_s,$$

where $\mathcal{E}_s^{\mathrm{k}} := \rho\widetilde{\mathbf{u}}_s^2/2$ is the classical kinetic energy, $\mathcal{E}_s^{\mathrm{t}} := \rho\langle\widetilde{\mathbf{c}}_s^2\rangle/2$ is the classical thermal energy, $\mathcal{E}_s := \mathcal{E}_s^{\mathrm{k}} + \mathcal{E}_s^{\mathrm{t}} = \rho_s\big(\frac{\widetilde{\mathbf{u}}_s^2}{2} + \frac{\langle\widetilde{\mathbf{c}}_s^2\rangle}{2}\big)$ is the gas energy, $\mathbf{q}_s := \rho_s\langle\widetilde{\mathbf{c}}_s\widetilde{\mathbf{c}}_s^2/2\rangle$ is the heat flux, i.e. the flux of heat energy through a boundary convected with the species velocity $\mathbf{u}_s$, and $Q_s := \int_{\widetilde{\mathbf{v}}} C_s^{\mathrm{m}}\widetilde{\mathbf{v}}^2/2$ is the rate of energy production (due to collisions with other species). That is,

$$\partial_t\mathcal{E} + \nabla\cdot(\mathbf{u}\mathcal{E} + \widetilde{\mathbf{u}}\cdot\mathbb{P} + \mathbf{q}) = \widetilde{\mathbf{J}}\cdot\mathbf{E} + \rho\widetilde{\mathbf{u}}\cdot\mathbf{g} + Q_s.$$

2.4 Convective derivative

Define $d_t^s := \partial_t + \mathbf{u}_s \cdot \nabla$, the *convective derivative for species s*. Verify the following Leibnitz rule for transport derivatives: $\eth_t(ab) = (\eth_t a)b + a(d_t b)$. So by mass balance ($\eth_t \rho = S_s$), we can write for any quantity b:

$$\eth_t(\rho b) = \rho d_t(b) + S_s b.$$

Since $\eth_t(\rho\widetilde{\mathbf{u}}) = \rho d_t \widetilde{\mathbf{u}} + S_a \widetilde{\mathbf{u}}$ and $\mathbf{A}_s = \widetilde{\mathbf{u}} S_s + \mathbf{R}_s$, we can write momentum balance as:

$$\rho_s d_t^s \widetilde{\mathbf{u}}_s + \nabla \cdot \mathbb{P}_s = \sigma_s \mathbf{E} + \mathbf{J}_s \times \mathbf{B} + \rho_s \mathbf{g} + \mathbf{R}_s.$$

2.5 Classical kinetic energy balance

To get an equation for kinetic energy balance, we simply dot $\widetilde{\mathbf{u}}$ with momentum balance. Since $\widetilde{\mathbf{u}} \cdot \rho d_t(\widetilde{\mathbf{u}}) = \rho d_t(\widetilde{\mathbf{u}}^2/2) = \eth_t(\rho\widetilde{\mathbf{u}}^2/2) - S_s \widetilde{\mathbf{u}}^2/2$, we get

$$\eth_t(\rho\widetilde{\mathbf{u}}^2/2) + \widetilde{\mathbf{u}} \cdot \nabla \cdot \mathbb{P} = \widetilde{\mathbf{J}} \cdot \mathbf{E} + \rho\widetilde{\mathbf{u}} \cdot \mathbf{g} + S_a \widetilde{\mathbf{u}}^2/2 + \widetilde{\mathbf{u}} \cdot \mathbf{R}_s.$$

2.6 Classical thermal energy balance

Recall energy balance:

$$\eth_t(\rho\widetilde{\mathbf{u}}^2/2 + \rho\langle\widetilde{\mathbf{c}}^2\rangle/2) + \nabla \cdot (\widetilde{\mathbf{u}} \cdot \mathbb{P} + \mathbf{q}) = \widetilde{\mathbf{J}} \cdot \mathbf{E} + \rho\widetilde{\mathbf{u}} \cdot \mathbf{g} + S_s \widetilde{\mathbf{u}}^2/2 + \widetilde{\mathbf{u}} \cdot \mathbf{R}_s + \mathcal{H}_s.$$

To obtain thermal energy balance, we subtract kinetic energy balance from energy balance. We get

$$\eth_t(\rho\langle\widetilde{\mathbf{c}}^2\rangle/2) + (\nabla\widetilde{\mathbf{u}}) : \mathbb{P} + \nabla \cdot \mathbf{q} = \mathcal{H}_s.$$

Observe that all macroscopic forces have disappeared, as one would expect.

2.7 $\chi = c^2\gamma$: relativistic balance of energy

Recall relativistic balance of energy:

$$\partial_t(\rho c^2\langle\gamma\rangle) + \nabla \cdot (\rho c^2\langle\mathbf{v}\gamma\rangle) = \rho\langle\mathbf{a} \cdot \mathbf{v}\rangle + \int_{\widetilde{\mathbf{v}}} C_s c^2 \gamma.$$

Relativistic kinetic energy is defined to be the energy minus the rest energy. Recall balance of rest energy:

$$\partial_t \rho c^2 + \nabla \cdot (\rho c^2 \mathbf{u}) = \left(\int_{\widetilde{\mathbf{v}}} C_s c^2 = c^2 S_s\right).$$

Subtracting the rest energy from the energy gives microscopic kinetic energy balance. (This is *not* the balance of macroscopic kinetic energy).

2.8 Relativistic kinetic energy balance

To get an equation for balance of bulk (macroscopic) kinetic energy, we begin with its definition: $\widetilde{\mathcal{E}^{\mathrm{k}}} := \rho(\widetilde{\gamma}(\widetilde{\mathbf{u}}) - 1)$. So

$$\begin{aligned}
\delta_t(\widetilde{\mathcal{E}^{\mathrm{k}}}) &= \delta_t(\rho(\widetilde{\gamma}(\widetilde{\mathbf{u}}) - 1)) \\
&= \rho d_t(\widetilde{\gamma}(\widetilde{\mathbf{u}}) - 1) \qquad + S_s(\widetilde{\gamma}(\widetilde{\mathbf{u}}) - 1) \\
&= \rho d_t(\widetilde{\mathbf{u}}) \cdot \nabla_{\widetilde{\mathbf{u}}} \cdot \widetilde{\gamma}(\widetilde{\mathbf{u}}) + S_s(\widetilde{\gamma}(\widetilde{\mathbf{u}}) - 1) \\
&= \rho d_t(\widetilde{\mathbf{u}}) \cdot \widetilde{\mathbf{u}}/\widetilde{\gamma}(\widetilde{\mathbf{u}}) \quad + S_s(\widetilde{\gamma}(\widetilde{\mathbf{u}}) - 1).
\end{aligned}$$

So we can get an equation for kinetic energy balance by taking the dot product of momentum conservation with $\frac{\widetilde{\mathbf{u}}}{\widetilde{\gamma}(\widetilde{\mathbf{u}})}$ (note that in general $\widetilde{\mathbf{u}} \neq \widetilde{\gamma}(\widetilde{\mathbf{u}})\mathbf{u}$, i.e., $\langle\widetilde{\mathbf{v}}\rangle = \langle\gamma(\mathbf{v})\mathbf{v}\rangle \neq \gamma(\langle\mathbf{v}\rangle)\langle\mathbf{v}\rangle$):

$$\begin{aligned}
&\Big(\rho d_t\widetilde{\mathbf{u}} + \nabla\cdot\mathbb{P} = \sigma\mathbf{E} + \mathbf{J}\times\mathbf{B} + \rho\mathbf{g} + \mathbf{R}_s\Big) \cdot \widetilde{\mathbf{u}}/\widetilde{\gamma}(\widetilde{\mathbf{u}}). \\
&\rho d_t(\widetilde{\gamma}(\widetilde{\mathbf{u}}) - 1) + \big(\widetilde{\mathbf{u}}/\widetilde{\gamma}(\widetilde{\mathbf{u}})\big)\cdot\nabla\cdot\mathbb{P} = (\sigma\mathbf{E} + \mathbf{J}\times\mathbf{B} + \rho\mathbf{g} + \mathbf{R}_s)\cdot\widetilde{\mathbf{u}}/\widetilde{\gamma}(\widetilde{\mathbf{u}}). \\
&\delta_t(\rho(\widetilde{\gamma}(\widetilde{\mathbf{u}}) - 1)) + \big(\widetilde{\mathbf{u}}/\widetilde{\gamma}(\widetilde{\mathbf{u}})\big)\cdot\nabla\cdot\mathbb{P} = (\sigma\mathbf{E} + \mathbf{J}\times\mathbf{B} + \rho\mathbf{g} + \mathbf{R}_s)\cdot\widetilde{\mathbf{u}}/\widetilde{\gamma}(\widetilde{\mathbf{u}}) + S_s(\widetilde{\gamma}(\widetilde{\mathbf{u}}) - 1).
\end{aligned}$$

3 Species constitutive equations

[From here unless otherwise indicated results hold only for the non-relativistic domain.]

To close the two-fluid system, it is necessary to supply constitutive relations for the pressure tensor, heat flux, and drag force in terms of the mass, momentum, and energy of each species.

3.1 Constitutive relation for pressure

Define the scalar pressure p to be one-third the trace of the pressure tensor. The translational thermal energy is half the trace of the pressure tensor, i.e.,

$$(\text{translational thermal energy}) \;= (3/2)p.$$

The simplest constitutive relation for the pressure tensor is to assume the thermodynamic equilibrium condition that the thermal energy is equally distributed among all degrees of freedom of the system. Assume that there are α degrees of freedom. Then

$$(\text{thermal energy}) \;= (\alpha/2)p.$$

The translational thermal energy density along a particular axis is half the pressure along that axis. So in equilibrium the pressure in all directions is equal, i.e., the pressure tensor equals the scalar pressure times the identity tensor:

$$\mathbb{P} = p\mathbb{I}.$$

In summary,

$$\mathcal{E} = (\alpha/2)p + (1/2)\rho u^2.$$

3.2 Constitutive equation for drag force

We derive a constitutive equation for the interspecies drag force, which we will need to supply a consitutive equation in Ohm's law for the resistance (due to collisions).

Current density is the flux rate of charge density. Momentum is the flux rate of mass density. Collisions conserve momentum, but they do not conserve current density. This implies that a constitutive relation for inter-species drag forces will be needed in the balance law for net current.

We seek a constitutive relation for $\mathbf{R}_s$, the resistive drag on species s due to collisions with other species that do not produce or destroy species s. Assuming that collisions do not destroy species s if and only if they do not destroy species p, Newton's third law gives $\mathbf{R}_{sp} = -\mathbf{R}_{ps}$.

We assume that drag force is jointly proportional to the densities of the interacting species and their difference in velocity: $\mathbf{R}_{sp} := \nu_{sp} n_s n_p (\mathbf{u}_p - \mathbf{u}_s)$, where n_s is the number density of species s. So $\nu_{sp} = \nu_{ps}$.

4 Entropy invariance

In the adiabatic case ($\mathbb{P} = p\mathbb{I}$, $\mathbf{q} = 0$, $\mathcal{H} = 0$, $S = 0$),

$$\frac{\alpha}{2}\eth_t(p) + (\nabla \cdot \mathbf{u})p = 0, \text{ i.e.,} \frac{\alpha}{2} d_t p + \frac{\alpha + 2}{2} p \nabla \cdot \mathbf{u}.$$

But mass conservation says that $\nabla \cdot \mathbf{u} = -d_t \ln \rho$, so

$$\begin{aligned} &d_t \ln p = \frac{\alpha + 2}{\alpha} d_t \ln \rho, \text{ i.e.,} \\ &d_t \ln(p\rho^{-\gamma}) = 0, \text{ where } \gamma := \frac{\alpha + 2}{\alpha}. \end{aligned}$$

5 Net fluid balance laws

[We resume noting relativistic corrections.]

The equations of magnetohydrodynamics (MHD) are a set of balance laws for the net mass, momentum, and energy (summed over all species) and for the magnetic field. A balance law for current, called Ohm's law, provides a constitutive relation that allows us to eliminate the electric field.

5.1 Balance of mass

Balance of (rest) mass per species is

$$\partial_t(\rho_s) + \nabla \cdot (\rho_s \mathbf{u}_s) = S_s,$$

where s is the species index, ρ_s is mass density, $\mathbf{u}_s$ is fluid velocity, and S_s is the rate of production of species s due to collisions.

This is a Lorentz-invariant equation which asserts that the four-divergence of the scalar $\left(\frac{\rho_s}{\gamma(\mathbf{u}_s)}\right)$ times the four-vector $\widetilde{\mathbf{u}}$ is the rate of production of proper mass, $S_s := \int_{\widetilde{\mathbf{v}}} C$:

$$\partial_{ct}\Big(\big(\frac{\rho_s}{\gamma(\mathbf{u}_s)}\big)(c\gamma(\mathbf{u}_s)) \Big) + \nabla \cdot \Big(\big(\frac{\rho_s}{\gamma(\mathbf{u}_s)}\big)\gamma(\mathbf{u}_s)\mathbf{u}_s \Big) = S_s.$$

Summing over all species gives

$$\partial_t(\rho) + \nabla \cdot (\rho \mathbf{u}) = 0,$$

where $\rho := \sum_s \rho_s$ is the total mass density, $(\rho\mathbf{u}) := \sum_s(\rho_s \mathbf{u}_s)$, the total mass flux, defines the net fluid velocity $\mathbf{u}$, and $S := \sum_s S_s$, the net rate of mass production, is assumed to be zero.

Define $\delta_t := a \mapsto \partial_t a + \nabla \cdot (\mathbf{u}a) = \partial_{x^\mu}\Big(\big(\frac{a}{\gamma(\mathbf{u})}\big)\big(\gamma(\mathbf{u})\mathbf{u}^\mu\big) \Big)$, the *transport* (or *bulk*) *derivative*. Then the mass balance law becomes

$$\delta_t(\rho) = 0.$$

[From here on results hold only for the non-relativistic domain.]

5.2 Balance of momentum

Recall balance of momentum for each species:

$$\partial_t(\rho_s \widetilde{\mathbf{u}}_s) + \nabla \cdot (\rho_s \mathbf{u}_s \widetilde{\mathbf{u}}_s) + \nabla \cdot \mathbb{P}_s = \sigma_s \mathbf{E} + \mathbf{J}_s \times \mathbf{B} + \rho_s \mathbf{g} + \mathbf{A}_s,$$

where $\mathbb{P}_s$ is the pressure tensor (the flux of momentum across a boundary convected with the species), σ_s is the charge density, $\mathbf{J}_s$ is the current density, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $\mathbf{g}$ is the gravitational field, and $\mathbf{A}_s$ is the resistive drag due to collisions with other species (regardless of whether the collisions do or do not create or destroy species s).

Summing over all species gives net balance of momentum:

$$\partial_t(\rho\widetilde{\mathbf{u}}) + \nabla \cdot (\rho \widetilde{\mathbf{u}}\widetilde{\mathbf{u}}) + \nabla \cdot \widetilde{\mathbb{P}} = \sigma \mathbf{E} + \mathbf{J} \times \mathbf{B} + \rho \mathbf{g},$$

where $\rho := \sum_s \rho_s$ is the total mass density, $(\rho\widetilde{\mathbf{u}}) := \sum_s \rho_s \widetilde{\mathbf{u}}_s$ defines the overall fluid velocity, $\mathbf{w}_s := \mathbf{u}_s - \mathbf{u}$ is the *diffusion* (or *drift*) *velocity of species s*, $\widetilde{\mathbf{w}}_s := \widetilde{\mathbf{u}}_s - \widetilde{\mathbf{u}}$ is the *proper diffusion* (or *drift*) *velocity of species s*, $\mathbb{P}^{\mathrm{d}} := \sum_a \mathbb{P}^{\mathrm{d}}_s := \sum_s \rho_s \mathbf{w}_s \widetilde{\mathbf{w}}_s$ is the total diffusion pressure tensor, $\mathbb{P}^{\mathrm{t}} := \sum_s \mathbb{P}_s$ is the total thermal pressure tensor, $\widetilde{\mathbb{P}} := \mathbb{P}^{\mathrm{t}} + \mathbb{P}^{\mathrm{d}}$ is the total pressure tensor (i.e., the total flux of momentum due to thermal and species diffusion across a boundary convected by global mean velocity), $\sigma := \sum_a \sigma_s$ is the net charge density, and $\mathbf{J} := \sum_s \mathbf{J}_s$ is the net current. Note that $\sum_s A_s = 0$, since momentum is conserved in all collisions (whether particles are converted to other types of species or not).

5.2.1 Convective derivative

Observe that the first two terms of the momentum balance equation are the transport derivative of the momentum, $\delta_t(\rho\mathbf{u})$.

As before, define $d_t := \partial_t + \mathbf{u} \cdot \nabla$, the *convective derivative* (for the bulk fluid). Verify/recall the Leibnitz rule for transport derivatives: $\delta_t(ab) = (\delta_t a)b + a(d_t b)$. So by the conservation of mass equation, $\delta_t \rho = 0$, we can pull ρ out of a transport derivative, thereby turning it into a convective derivative: for example, $\delta_t(\rho\mathbf{u}) = \rho d_t(\mathbf{u})$, i.e., $\partial_t(\rho\mathbf{u}) + \nabla \cdot (\rho\mathbf{u}\mathbf{u}) = \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u})$.

5.2.2 Conservation of momentum

To recast momentum balance as a *conservation* law, we use Maxwell's equations to recast the source term as the time derivative of the momentum of the magnetic field plus the divergence of a stress tensor:

$$
\begin{aligned}
&\sigma\mathbf{E} + \mathbf{J}\times\mathbf{B} \\
&= \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + \big(\mu_0^{-1}\nabla\times\mathbf{B} - \epsilon_0\partial_t\mathbf{E}\big)\times\mathbf{B} \\
&= \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + \mu_0^{-1}(\nabla\times\mathbf{B})\times\mathbf{B} - \epsilon_0(\partial_t\mathbf{E})\times\mathbf{B} \\
&= \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + \mu_0^{-1}\big(\mathbf{B}\cdot\nabla\mathbf{B} - \nabla B^2/2\big) - \epsilon_0\big[\partial_t(\mathbf{E}\times\mathbf{B}) - \mathbf{E}\times\partial_t\mathbf{B}\big] \\
&= -\epsilon_0\partial_t(\mathbf{E}\times\mathbf{B}) + \mu_0^{-1}\big(\mathbf{B}\cdot\nabla\mathbf{B} - \nabla B^2/2\big) - \epsilon_0\mathbf{E}\times(\nabla\times\mathbf{E}) + \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} \\
&= -\epsilon_0\partial_t(\mathbf{E}\times\mathbf{B}) + \mu_0^{-1}\big(\mathbf{B}\cdot\nabla\mathbf{B} - \nabla B^2/2\big) + \epsilon_0\Big(\mathbf{E}\cdot\nabla\mathbf{E} - (\nabla\mathbf{E})\cdot\mathbf{E} + (\nabla\cdot\mathbf{E})\mathbf{E}\Big) \\
&= -\epsilon_0\partial_t(\mathbf{E}\times\mathbf{B}) + \mu_0^{-1}\nabla\cdot\big(\mathbf{BB} - \mathbb{I}B^2/2\big) + \epsilon_0\big(\nabla\cdot(\mathbf{EE}) - \nabla(E^2/2)\big) \\
&= -\partial_t(\epsilon_0\mathbf{E}\times\mathbf{B}) + \nabla\cdot\Big(\mu_0^{-1}\big(\mathbf{BB} - \mathbb{I}B^2/2\big) + \epsilon_0\big(\mathbf{EE} - \mathbb{I}E^2/2\big)\Big) \\
&= -\partial_t(\mathbf{S}/c^2) + \nabla\cdot\mathbb{T}^{EB},
\end{aligned}
$$

where $\mathbf{S} := \mu_0^{-1}\mathbf{E}\times\mathbf{B}$ is the Poynting vector, $\mathbf{S}/c^2$ is the momentum of the electromagnetic field, and $\mathbb{T}^{EB} := \mu_0^{-1}\big(\mathbf{BB} - \mathbb{I}B^2/2\big) + \epsilon_0\big(\mathbf{EE} - \mathbb{I}E^2/2\big)$ is the electromagnetic stress tensor.

Writing $\mathbf{g} = -\nabla\chi$, conservation of momentum reads

$$\delta_t(\rho\mathbf{u}) + \partial_t(\epsilon_0\mathbf{E}\times\mathbf{B}) + \nabla\cdot\tilde{\mathbb{P}} + \nabla\chi = \nabla\cdot\mathbb{T}^{EB}.$$

The one-fluid theory eliminates the electric field by neglecting the displacement current $\partial_t\mathbf{E}$ and second-order terms in $\mathbf{E}$. This effectively sets $\mathbf{E} = 0$ in the momentum equation, so you can delete all the colored text, and one-fluid conservation of momentum reads

$$\delta_t(\rho\mathbf{u}) + \nabla\cdot\tilde{\mathbb{P}} + \nabla\cdot\mu_0^{-1}\big(\mathbb{I}B^2/2 - \mathbf{BB}\big) + \nabla\chi = 0. \tag{5.1}$$

5.3 Balance of current (Ohm's law)

The version of Ohm's law used is what determines the model of MHD: ideal, resistive, Hall, or extended. Ohm's law specifies the evolution of electrical current in a plasma (in response to electromagnetic field) in terms of the quantities of the one-fluid plasma model (MHD). Recall Faraday's law: $\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0$. Ohm's law is a balance law for the current. Ohm's law is a constitutive relation that we use to eliminate $\mathbf{E}$ from Faraday's law. The primary assumptions used to derive the generalized Ohm's law are quasineutrality and that there are only two species, ions and electrons.

Each momentum balance equation becomes a current balance law if we multiply it by the ratio of charge to mass:

$$\partial_t(\sigma_s\mathbf{u}_s) + \nabla\cdot(\sigma_s\mathbf{u}_s\mathbf{u}_s) + \nabla\cdot\mathbb{P}_s^q = \frac{q_s}{m_s}(\sigma_s\mathbf{E} + \mathbf{J}_s\times\mathbf{B}) + \sigma_s\mathbf{g} + \mathbf{A}_s^q,$$

where (for species s) $\mathbb{P}_s^q := \frac{q_s}{m_s}\mathbb{P}_s$ is the *electrokinetic pressure tensor*, and $\mathbf{A}_s^q := \frac{q_s}{m_s}\mathbf{A}_s$ is the production of current due to collisions.

Summing over all species gives net current balance. The net current is $\sum_s \sigma_s\mathbf{u}_s = \sum_s \mathbf{J}_s =: \mathbf{J}$. Define also $\mathbf{J}' := \sum_s \mathbf{J}_s' := \sum_s \sigma_s\mathbf{w}_s$, the current density in the reference frame of the fluid. Observe that

$\mathbf{J}' = \sum_s \sigma_s(\mathbf{u}_s - \mathbf{u}) = \mathbf{J} - \sigma\mathbf{u}$. Then the current fluxes sum as:

$$
\begin{aligned}
&\sum_s \sigma_s \mathbf{u}_s \mathbf{u}_s = \sum_s \sigma_s (\mathbf{u} + \mathbf{w}_s)(\mathbf{u} + \mathbf{w}_s) \\
&= \sum_s \big(\sigma_s \mathbf{u}\mathbf{u} + \sigma_s \mathbf{w}_s \mathbf{u} + \sigma_s \mathbf{u}\mathbf{w}_s + \sigma_s \mathbf{w}_s \mathbf{w}_s\big) \\
&= \sigma \mathbf{u}\mathbf{u} + \mathbf{J}'\mathbf{u} + \mathbf{u}\mathbf{J}' + \sum_s \sigma_s \mathbf{w}_s \mathbf{w}_s \\
&= \mathbf{J}\mathbf{u} + \mathbf{u}\mathbf{J} - \sigma\mathbf{u}\mathbf{u} + \sum_s \sigma_s \mathbf{w}_s \mathbf{w}_s.
\end{aligned}
$$

Define $\mathbb{P}_s^{\mathrm{qd}} := \sigma_s \mathbf{w}_s \mathbf{w}_s = \mathbf{J}_s' \mathbf{J}_s' / \sigma_s$ $(= \sum_s \frac{q_s}{m_s} \mathbb{P}_s^{\mathrm{d}})$, the *diffusion electrokinetic pressure tensor for species* s, and $\mathbb{P}^{\mathrm{qd}} := \sum_s \mathbb{P}_s^{\mathrm{qd}} = \sum_s \sigma_s \mathbf{w}_s \mathbf{w}_s$, the *total diffusion electrokinetic pressure tensor*. We also define $\mathbb{P}^{\mathrm{q}} := \sum_s \mathbb{P}_s^{\mathrm{q}}$, the *total thermal electrokinetic pressure tensor* . So total current balance reads

$$
\partial_t \mathbf{J} + \nabla \cdot \Big(\mathbf{J}\mathbf{u} + \mathbf{u}\mathbf{J} - \sigma\mathbf{u}\mathbf{u} + \sum_s \mathbf{J}_s' \mathbf{J}_s' / \sigma_s\Big) + \nabla \cdot \mathbb{P}^{\mathrm{q}} = \sum_s \frac{q_s}{m_s} (\sigma_s \mathbf{E} + \mathbf{J}_s \times \mathbf{B}) + \sigma \mathbf{g} + \mathbf{A}^q,
$$

where $\mathbf{A}^q := \sum_s \mathbf{A}_s^q$.

Ohm's law is current balance solved for $\mathbf{E}$. For this to give system closure, all two-fluid quantities in Ohm's law need to be expressed in terms of one-fluid quantities, and it is necessary to supply a constitutive relation for $\mathbf{A}^q$. We will do so for the case of a quasineutral two-species plasma with negligible displacement current and linear collisional drag force.

5.4 Balance of energy

Recall balance of energy for each species:

$$
\partial_t \mathcal{E}_s + \nabla \cdot (\mathbf{u}_s \mathcal{E}_s + \mathbf{u}_s \cdot \mathbb{P}_s + \mathbf{q}_s) = \mathbf{J}_s \cdot \mathbf{E} + \rho_s \mathbf{u}_s \cdot \mathbf{g} + Q_s.
$$

Summing over all species gives total energy balance.

Recall the decomposition of the species velocity as $\mathbf{u}_s =: \mathbf{u} + \mathbf{w}_s$, the sum of the net fluid velocity and the species diffusion velocity, where $\sum_s \rho_s \mathbf{w}_s = 0$. Define ${\mathcal{E}^{\mathrm{k}}}'_s := \rho_s w_s^2/2$, the kinetic energy of species s in the reference frame of the net fluid flow, and define $\mathcal{E}_s' := \mathcal{E}_s^{\mathrm{t}} + {\mathcal{E}^{\mathrm{k}}}'_s$, the gas-dynamic energy of species s in the reference frame of the net fluid flow.

We now show that the nonlinear term gives rise to higher-order terms which are naturally absorbed into the higher-order moment, the heat flux. The total gas-dynamic energy is $\mathcal{E} := \sum_s \mathcal{E}_s$. For the nonlinear flux terms, define $\mathbf{q} := \sum_s \mathbf{q}_s$. We compute the nonlinear flux terms:

$$
\begin{aligned}
&\sum_s \mathbf{u}_s \mathcal{E}_s && + \sum_s \mathbf{u}_s \cdot \mathbb{P}_s && + \sum_s \mathbf{q}_s \\
&= \sum_s \mathbf{u}\mathcal{E}_s + \sum_s \mathbf{w}_s \mathcal{E}_s && + \sum_s \mathbf{u} \cdot \mathbb{P}_s + \sum_s \mathbf{w}_s \cdot \mathbb{P}_s && + \mathbf{q} \\
&= \mathbf{u}\mathcal{E} + \sum_s \mathbf{w}_s (\mathcal{E}_s^{\mathrm{t}} + \rho_s u_s^2/2) && + \mathbf{u} \cdot \mathbb{P} + \sum_s \mathbf{w}_s \cdot \mathbb{P}_s && + \mathbf{q} \\
&= \mathbf{u}\mathcal{E} + \sum_s \mathbf{w}_s \underbrace{(\mathcal{E}_s^{\mathrm{t}} + \rho_s w_s^2/2)}_{\mathcal{E}_s'} + \mathbf{u} \cdot \underbrace{\sum_s \rho_s \mathbf{w}_s \mathbf{w}_s}_{\mathbb{P}^{\mathrm{d}}} && + \mathbf{u} \cdot \mathbb{P} + \sum_s \mathbf{w}_s \cdot \mathbb{P}_s && + \mathbf{q}.
\end{aligned}
$$

Reordering the terms to absorb the "bad stuff" into the pressure tensor and heat flux, this is

$$\mathbf{u}\mathcal{E} + \underbrace{\mathbf{u}\cdot\big(\mathbb{P} + \underbrace{\sum_s \rho_s\mathbf{w}_s\mathbf{w}_s}_{\mathbb{P}^{\mathrm{d}}}\big)}_{\tilde{\mathbb{P}}} + \underbrace{\big(\mathbf{q} + \sum_s \mathbf{w}_s\cdot\mathbb{P}_s + \sum_s \mathbf{w}_s\,\underbrace{(\mathcal{E}^{\mathrm{t}}_s + \rho_s w_s^2/2)}_{\mathcal{E}'_s}\big)}_{\tilde{\mathbf{q}}}.$$

So we have formulated net energy balance in the familiar form,

$$\partial_t\mathcal{E} + \nabla\cdot(\mathbf{u}\mathcal{E} + \mathbf{u}\cdot\tilde{\mathbb{P}} + \tilde{\mathbf{q}}) = \mathbf{J}\cdot\mathbf{E} + \rho\mathbf{u}\cdot\mathbf{g},$$

where $\mathcal{E} := \sum_s \mathcal{E}_s$ is the total gas-dynamic energy, $\tilde{\mathbb{P}} := \mathbb{P} + \mathbb{P}^{\mathrm{d}} := \sum_s \mathbb{P}_s + \sum_s \rho_s\mathbf{w}_s\mathbf{w}_s$ is the total thermal and diffusion pressure, and $\tilde{\mathbf{q}} := \mathbf{q} + \mathbf{q}^{\mathrm{d}} + \mathcal{W}^{\mathrm{d}} := \sum_s \mathbf{q}_s + \sum_s \mathbf{w}_s\mathcal{E}'_s + \sum_s \mathbf{w}_s\cdot\mathbb{P}_s$ is the total heat flux due to thermal diffusion, species diffusion, and work per species.

5.4.1 Conservation of energy

To recast this as a *conservation* law, we use Ampere's and Faraday's laws to recast the opposite of the source term as the time derivative of something we will call the energy of the magnetic field plus the divergence of something we will call the electromagnetic energy flux.

$$\begin{aligned}
&-\mathbf{J}\cdot\mathbf{E} = (\epsilon_0\partial_t\mathbf{E} - \mu_0^{-1}\nabla\times\mathbf{B})\cdot\mathbf{E}\\
&= \epsilon_0\partial_t(\tfrac{1}{2}E^2) - \mu_0^{-1}\mathbf{E}\cdot\nabla\times\mathbf{B}\\
&= \epsilon_0\partial_t(\tfrac{1}{2}E^2) - \mu_0^{-1}(\mathbf{B}\cdot\nabla\times\mathbf{E} - \nabla\cdot(\mathbf{E}\times\mathbf{B}))\\
&= \epsilon_0\partial_t(\tfrac{1}{2}E^2) + \mu_0^{-1}\partial_t(\tfrac{1}{2}B^2) + \mu_0^{-1}\nabla\cdot(\mathbf{E}\times\mathbf{B})\\
&= \partial_t\big(\underbrace{\epsilon_0(\tfrac{1}{2}E^2) + \mu_0^{-1}(\tfrac{1}{2}B^2)}_{\text{Call } \mathcal{E}^{\mathrm{f}}}\big) + \nabla\cdot\big(\underbrace{\tfrac{1}{\mu_0}\mathbf{E}\times\mathbf{B}}_{\text{Call } \mathbf{S}}\big).
\end{aligned}$$

Writing $\mathbf{g} = -\nabla\chi$, and observing that $\rho\mathbf{u}\cdot\nabla\chi = \nabla\cdot(\rho\mathbf{u}\chi) - \nabla\cdot(\rho\mathbf{u})\chi = \nabla\cdot(\rho\mathbf{u}\chi) + (\partial_t\rho)\chi = \nabla\cdot(\mathbf{u}(\rho\chi) + \partial_t(\rho\chi) - \rho\cancel{\partial_t\chi}$, conservation of energy reads

$$\partial_t(\mathcal{E} + \rho\chi) + \partial_t\big(\epsilon_0 E^2/2 + \frac{B^2}{2\mu_0}\big) + \nabla\cdot(\mathbf{u}(\mathcal{E}+\rho\chi) + \mathbf{u}\cdot\tilde{\mathbb{P}} + \tilde{\mathbf{q}}) + \nabla\cdot\big(\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\big) = 0. \tag{5.2}$$

In MHD we neglect $\epsilon_0 E^2/2$. We will also neglect $\rho\chi$.

6 Net fluid balance laws for two species

6.1 Expressing two-fluid quantities in terms of one-fluid quantities

We assume that there are only two species, ions (i) and electrons (e). Assume singly charged ions. We can express $\mathbf{J}_i$ and $\mathbf{J}_e$ in terms of one-fluid quantities using the definitions of the net first moments, the charge and mass, $\mathbf{J}$ and $(\rho\mathbf{u})$:

$$\left\{\begin{matrix}\mathbf{J} & = & \mathbf{J}_i + \mathbf{J}_e,\\ \rho\mathbf{u} & = & (\rho_i\mathbf{u}_i) + (\rho_e\mathbf{u}_e)\end{matrix}\right\}, \text{ i.e., } \begin{pmatrix}\mathbf{J}\\ \rho\mathbf{u}\end{pmatrix} = \begin{bmatrix}1 & 1\\ \tilde{m}_i & -\tilde{m}_e\end{bmatrix}\begin{pmatrix}\mathbf{J}_i\\ \mathbf{J}_e\end{pmatrix},$$

where $\tilde{m}_i := \frac{m_i}{e}$ and $\tilde{m}_e := \frac{m_e}{e}$.

Solving this linear system for $\mathbf{J}_i$ and $\mathbf{J}_e$ gives

$$\begin{pmatrix}\mathbf{J}_i\\ \mathbf{J}_e\end{pmatrix} = \frac{1}{\tilde{m}_i+\tilde{m}_e}\begin{bmatrix}\tilde{m}_e & 1\\ \tilde{m}_i & -1\end{bmatrix}\begin{pmatrix}\mathbf{J}\\ \rho\mathbf{u}\end{pmatrix} = \begin{pmatrix}\tilde{m}/\tilde{m}_i\\ \tilde{m}/\tilde{m}_e\end{pmatrix}\mathbf{J} + \begin{pmatrix}1\\ -1\end{pmatrix}\frac{\rho\mathbf{u}}{\tilde{m}_i+\tilde{m}_e} = \begin{pmatrix}(\tilde{m}/\tilde{m}_i)(\mathbf{J}+\rho\mathbf{u}/\tilde{m}_e)\\ (\tilde{m}/\tilde{m}_e)(\mathbf{J}-\rho\mathbf{u}/\tilde{m}_i)\end{pmatrix},$$

where the *reduced mass* μ, defined by

$$\mu^{-1} := m_i^{-1} + m_e^{-1} \approx m_e^{-1}, \qquad \text{i.e.,} \qquad \mu = \frac{m_i m_e}{m_i + m_e},$$

is slightly smaller than the mass of an electron, and where $\tilde{m} := \frac{m}{e}$ (so $\frac{\tilde{m}}{\tilde{m}_s} = \frac{\mu}{m_s}$).

Transforming to the reference frame of the fluid (i.e. expressing in terms of relative velocities via $\mathbf{u}_s = \mathbf{u} + \mathbf{w}_s$) and multiplying the momentum equation by e, this becomes

$$\left\{\begin{matrix}\mathbf{J}' &=& \mathbf{J}'_i + \mathbf{J}'_e,\\ 0 &=& \tilde{m}_i\mathbf{J}'_i - \tilde{m}_e\mathbf{J}'_e\end{matrix}\right\}, \text{ i.e., } \begin{pmatrix}\mathbf{J}'\\ 0\end{pmatrix} = \begin{bmatrix}1 & 1\\ \tilde{m}_i & -\tilde{m}_e\end{bmatrix}\begin{pmatrix}\mathbf{J}'_i\\ \mathbf{J}'_e\end{pmatrix},$$

which has solution

$$\begin{pmatrix}\mathbf{J}'_i\\ \mathbf{J}'_e\end{pmatrix} = \frac{-1}{\tilde{m}_i+\tilde{m}_e}\begin{bmatrix}-\tilde{m}_e & 1\\ -\tilde{m}_i & 1\end{bmatrix}\begin{pmatrix}\mathbf{J}'\\ 0\end{pmatrix} = \frac{1}{\tilde{m}_i+\tilde{m}_e}\begin{pmatrix}\tilde{m}_e\\ \tilde{m}_i\end{pmatrix}\mathbf{J}' = \begin{pmatrix}\tilde{m}/\tilde{m}_i\\ \tilde{m}/\tilde{m}_e\end{pmatrix}\mathbf{J}'.$$

We could have obtained the current in the rest frame from the current in the fluid frame (or vice versa), using the fact that $\mathbf{J}_s = \mathbf{J}'_s + \sigma_s\mathbf{u}$, as follows. We can express σ_i and σ_e in terms of one-fluid quantities using the definitions of the net zeroth moments of the charge and mass, σ and ρ:

$$\left\{\begin{matrix}\sigma &=& \sigma_i + \sigma_e,\\ \rho &=& \rho_i + \rho_e\end{matrix}\right\}, \text{ i.e., } \begin{pmatrix}\sigma\\ e\rho\end{pmatrix} = \begin{bmatrix}1 & 1\\ \tilde{m}_i & -\tilde{m}_e\end{bmatrix}\begin{pmatrix}\sigma_i\\ \sigma_e\end{pmatrix}.$$

Solving this linear system for σ_i and σ_e gives

$$\begin{pmatrix}\sigma_i\\ \sigma_e\end{pmatrix} = \frac{1}{\tilde{m}_i+\tilde{m}_e}\begin{bmatrix}\tilde{m}_e & 1\\ \tilde{m}_i & -1\end{bmatrix}\begin{pmatrix}\sigma\\ \rho\end{pmatrix} = \begin{pmatrix}\tilde{m}/\tilde{m}_i\\ \tilde{m}/\tilde{m}_e\end{pmatrix}\sigma + \begin{pmatrix}1\\ -1\end{pmatrix}\frac{\rho}{\tilde{m}_i+\tilde{m}_e} = \begin{pmatrix}(\tilde{m}/\tilde{m}_i)(\sigma+\rho/\tilde{m}_e)\\ (\tilde{m}/\tilde{m}_e)(\sigma-\rho/\tilde{m}_i)\end{pmatrix}.$$

In summary,

$$\begin{aligned} en_i\mathbf{w}_i &= \mathbf{J}'_i = \frac{\tilde{m}}{\tilde{m}_i}(\mathbf{J}-\sigma\mathbf{u}), & -en_e\mathbf{w}_e &= \mathbf{J}'_e = \frac{\tilde{m}}{\tilde{m}_e}(\mathbf{J}-\sigma\mathbf{u}),\\ en_i\mathbf{u}_i &= \mathbf{J}_i = \frac{\tilde{m}}{\tilde{m}_i}(\mathbf{J}+\frac{\rho\mathbf{u}}{\tilde{m}_e}), & -en_e\mathbf{u}_e &= \mathbf{J}_e = \frac{\tilde{m}}{\tilde{m}_e}(\mathbf{J}-\frac{\rho\mathbf{u}}{\tilde{m}_i}),\\ en_i &= \sigma_i = \frac{\tilde{m}}{\tilde{m}_i}(\sigma+\frac{\rho}{\tilde{m}_e}), & -en_e &= \sigma_e = \frac{\tilde{m}}{\tilde{m}_e}(\sigma-\frac{\rho}{\tilde{m}_i}). \end{aligned}$$

Dividing the top two pairs of equations by the bottom pair gives us formulas for the species velocities:

$$\begin{aligned} \mathbf{w}_i &= \frac{\mathbf{J}'_i}{\sigma_i} = \frac{\mathbf{J}-\sigma\mathbf{u}}{\sigma+\frac{\rho}{\tilde{m}_e}} \approx \frac{\tilde{m}_e\mathbf{J}}{\rho} \text{ if } \sigma \approx 0, & \mathbf{w}_e &= \frac{\mathbf{J}'_e}{\sigma_e} = \frac{\mathbf{J}-\sigma\mathbf{u}}{\sigma-\frac{\rho}{\tilde{m}_i}} \approx -\frac{\tilde{m}_i\mathbf{J}}{\rho} \text{ if } \sigma \approx 0,\\ \mathbf{u}_i &= \frac{\mathbf{J}_i}{\sigma_i} = \frac{\mathbf{J}+\frac{\rho\mathbf{u}}{\tilde{m}_e}}{\sigma+\frac{\rho}{\tilde{m}_e}} = \mathbf{u}+\mathbf{w}_i, & \mathbf{u}_e &= \frac{\mathbf{J}_e}{\sigma_e} = \frac{\mathbf{J}+\frac{\rho\mathbf{u}}{\tilde{m}_i}}{\sigma+\frac{\rho}{\tilde{m}_i}} = \mathbf{u}+\mathbf{w}_e. \end{aligned}$$

So we have formulas for the zeroth and first moments of each species in terms of two-fluid variables.

We now substitute to obtain expressions for the coefficients of the Lorentz force law in terms of one-fluid variables.

$$\sum_s \frac{q_s}{m_s}\sigma_a = \frac{\sigma_i}{\tilde{m}_i} + \frac{\sigma_e}{\tilde{m}_e} = \frac{1}{\tilde{m}_i}\frac{\tilde{m}}{\tilde{m}_i}\Big(\sigma + \frac{\rho}{\tilde{m}_e}\Big) + \frac{1}{\tilde{m}_e}\frac{\tilde{m}}{\tilde{m}_e}\Big(\sigma - \frac{\rho}{\tilde{m}_i}\Big)$$
$$= \Big(\frac{1}{\tilde{m}_i} - \frac{1}{\tilde{m}_e}\Big)\sigma + \frac{\rho}{\tilde{m}_i\tilde{m}_e} \approx \frac{e^2 n}{\mu} \text{ if } \sigma \approx 0.$$

Similarly (by making the replacements $\sigma \mapsto \mathbf{J}$ and $\rho \mapsto \rho\mathbf{u}$),

$$\sum_s \frac{q_s}{m_s}\mathbf{J}_a = \Big(\frac{1}{\tilde{m}_i} - \frac{1}{\tilde{m}_e}\Big)\mathbf{J} + \frac{\rho\mathbf{u}}{\tilde{m}_i\tilde{m}_e} \approx e\Big(\frac{1}{m_i} - \frac{1}{m_e}\Big)\mathbf{J} + \frac{e^2 n}{\mu}\mathbf{u} \text{ if } \sigma \approx 0.$$

The general expression for the diffusion pressure tensor in terms of 1-fluid quantities doesn't seem to simplify to anything nicer, so we compute it here only in the quasineutral case. Since $\mathbf{J}' = \mathbf{J} - \sigma\mathbf{u}$, $\sigma \approx 0$, says that $\mathbf{J}' \approx \mathbf{J}$. So

$$\mathbb{P}^{\mathrm{qd}} = \sum_s \sigma_s \mathbf{w}_s\mathbf{w}_s = \sum_s \mathbf{J}'_s\mathbf{w}'_s \approx \frac{\tilde{m}_e^2 - \tilde{m}_i^2}{\tilde{m}_i + \tilde{m}_e}\frac{\mathbf{JJ}}{\rho} = \frac{\tilde{m}_e^2 - \tilde{m}_i^2}{\tilde{m}_i + \tilde{m}_e}\frac{\mathbf{JJ}}{\rho}$$
$$= (\tilde{m}_e - \tilde{m}_i)\frac{\mathbf{JJ}}{\rho}, \approx -\frac{\mathbf{JJ}}{ne} \quad \text{if} \quad m_e \ll m_i.$$

Finally, we derive a constitutive relation for the resistance, i.e., the rate of production of current due to collisions. Assume that there are no collisions which annihilate either species. So $S_s = 0$, and $\mathbf{A}_s = \mathbf{R}_s$. So $\mathbf{A}^q = \sum_s \mathbf{R}_s^q = \sum_s \sum_{p\neq s} \frac{q_s}{m_s}\nu_{sp} n_s n_p(\mathbf{u}_p - \mathbf{u}_s) = -n_i n_e \nu_{ie}(\frac{q_i}{m_i} - \frac{q_e}{m_e})(\mathbf{u}_i - \mathbf{u}_e)$. Assuming singly charged ions,

$$\mathbf{A}^q = -\frac{e n_i n_e \nu_{ei}}{\mu}(\mathbf{u}_i - \mathbf{u}_e) = -\mu^{-1}\nu_{ei}(n_e\mathbf{J}_i + n_i\mathbf{J}_e). \tag{6.1}$$

So in the case of quasineutrality ($n_i \approx n_e \approx (n_i + n_e)/2 =: n$),

$$\mathbf{A}^q \approx \frac{-n\nu_{ei}}{\mu}\mathbf{J}.$$

The negative sign indicates that the resistive drag acts to decrease the magnitude on the current, as expected.

6.2 Ohm's law: Current balance expressed in terms of one-fluid quantities

So the assumption of quasineutrality yields the generalized Ohm's law,

$$\partial_t\mathbf{J} + \nabla\cdot\Big(\mathbf{Ju} + \mathbf{uJ} + (\frac{m_e - m_i}{m_i + m_e})\frac{\mathbf{JJ}}{ne}\Big) + \nabla\cdot\mathbb{P}^q = \frac{e^2 n}{\mu}\Big(\mathbf{E} + \mathbf{u}\times\mathbf{B}\Big) + e\Big(\frac{1}{m_i} - \frac{1}{m_e}\Big)\mathbf{J}\times\mathbf{B} - \frac{n\nu_{ei}}{\mu}\mathbf{J}.$$

where recall that $\mathbb{P}^q := e\left(\frac{\mathbb{P}_i}{m_i} + \frac{\mathbb{P}_e}{m_e}\right)$. Solving for $\mathbf{E}$ gives

$$\mathbf{E} = \mathbf{B}\times\mathbf{u} + \frac{\nu}{e^2}\mathbf{J} + \underbrace{\frac{\mu}{en}\Big(\frac{1}{m_e} - \frac{1}{m_i}\Big)\mathbf{J}\times\mathbf{B}}_{\text{Hall term}} + \frac{\mu}{e^2 n}\Big[\nabla\cdot\mathbb{P}^q + \partial_t\mathbf{J} + \nabla\cdot\Big(\mathbf{Ju} + \mathbf{uJ} + (\frac{m_e - m_i}{m_i + m_e})\frac{\mathbf{JJ}}{ne}\Big)\Big]. \tag{6.2}$$

Assuming negligible electron mass ($m_e \ll m_i$) yields:

$$\mathbf{E} = \mathbf{B} \times \mathbf{u} + \frac{\nu}{e^2}\mathbf{J} + \frac{1}{en}\mathbf{J} \times \mathbf{B} - \frac{1}{en}\nabla \cdot \mathbb{P}_e + \frac{m_e}{e^2 n}\Big[\partial_t \mathbf{J} + \nabla \cdot (\mathbf{J}\mathbf{u} + \mathbf{u}\mathbf{J} - \frac{\mathbf{J}\mathbf{J}}{ne})\Big]. \tag{6.3}$$

But in the case of an electron-positron pair plasma ($m_e = m_i = 2\mu$),

$$\mathbf{E} = \mathbf{B} \times \mathbf{u} + \frac{\nu}{e^2}\mathbf{J} + \frac{1}{2en}\nabla \cdot (\mathbb{P}_i - \mathbb{P}_e) + \frac{\mu}{e^2 n}\Big[\partial_t \mathbf{J} + \nabla \cdot \Big(\mathbf{J}\mathbf{u} + \mathbf{u}\mathbf{J}\Big)\Big].$$

6.3 Balance of momentum for two species

Assuming two species allows us to write out the total diffusion pressure tensor in terms of bulk quantities. For simplicity we assume quasineutrality. Then

$$\mathbf{w}_i - \mathbf{w}_e = \frac{\mathbf{J}}{ne}.$$

Also from quasineutrality,

$$\mathbf{w}_i = \frac{\mu}{m_i}\frac{\mathbf{J}}{en}, \qquad \text{i.e.,} \qquad \rho_i \mathbf{w}_i = \frac{\mu}{e}\mathbf{J}, \text{ and}$$
$$\mathbf{w}_e = -\frac{\mu}{m_e}\frac{\mathbf{J}}{en}, \qquad \text{i.e.,} \qquad \rho_e \mathbf{w}_e = -\frac{\mu}{e}\mathbf{J}.$$

So

$$\begin{aligned}\mathbb{P}^{\mathrm{d}} &= \sum_s \rho_s \mathbf{w}_s \mathbf{w}_s = \frac{\mu}{e}\mathbf{J}(\mathbf{w}_i - \mathbf{w}_e) \\ &= \frac{\mu}{e^2 n}\mathbf{J}\mathbf{J}.\end{aligned}$$

So momentum equation (5.1) is

$$\bar{\partial}_t(\rho\mathbf{u}) + \nabla \cdot \mathbb{P} + \nabla \cdot \Big(\frac{\mu}{e^2 n}\mathbf{J}\mathbf{J}\Big) + \nabla \cdot \mu_0^{-1}\big(\mathbb{I}B^2/2 - \mathbf{B}\mathbf{B}\big) + \nabla\chi = 0.$$

(Generally $\mathbf{JJ}$ is neglected as a second-order term.)

7 Mapping between MHD and 2-fluid variables

The mapping from 2-fluid to 1-fluid variables is straightforward. For conserved variables:

$$\begin{aligned}\rho &= \rho_i + \rho_e \\ (\rho\mathbf{u}) &= (\rho\mathbf{u})_i + (\rho\mathbf{u})_e \\ \mathcal{E} &= \mathcal{E}_i + \mathcal{E}_e \\ \mathbf{B} &= \mathbf{B}\end{aligned}$$

Also,

$$\begin{aligned}&\mathbb{P} \approxeq \mathbb{P}_i + \mathbb{P}_e, \\ &\mathbf{J} \approxeq \mu_0^{-1}\nabla \times \mathbf{B}.\end{aligned}$$

Since the mapping from 2-fluid to 1-fluid variables is not one-to-one, the mapping from 1-fluid to 2-fluid variables requires auxiliary information or assumptions. The four specific things that we must specify are (1) the ratio of particle number densities (i.e. quasineutrality) and (2) the ratio of temperatures between the two species, and constitutive relations for the (3) current (typically by neglect of displacement current) and (4) electric field (using some form of Ohm's law).

The MHD assumption of quasineutrality says that the ratio of particle number densities is one. So the contribution of each species to the density is:

$$\begin{aligned} n &= \rho/(m_i + m_e), \\ \rho_i &= \frac{m_i}{m_i + m_e}\rho, \\ \rho_e &= \frac{m_e}{m_i + m_e}\rho, \end{aligned}$$

The MHD magnetostatic assumption gives the current:

$$\mathbf{J} = \mu_0^{-1}\nabla \times \mathbf{B}.$$

Combining this with quasineutrality gives the current and thus the velocity of each species:

$$\begin{aligned} \mathbf{J}_i &= \frac{m_e}{m_i + m_e}\mathbf{J}, \\ \mathbf{J}_e &= \frac{m_i}{m_i + m_e}\mathbf{J}, \\ \mathbf{w}_i &= \quad \mathbf{J}_i/(ne) = \quad \tilde{m}_e \mathbf{J}/\rho, \\ \mathbf{w}_e &= -\mathbf{J}_e/(ne) = -\tilde{m}_i \mathbf{J}/\rho, \\ \mathbf{u}_i &= \mathbf{u} + \mathbf{w}_i, \\ \mathbf{u}_e &= \mathbf{u} + \mathbf{w}_e, \end{aligned}$$

Assuming scalar pressure, combining the ideal gas laws $p_s = n_s(kT_s)$ with quasineutrality ($n_s = n$), and using $p = p_i + p_e$ gives the contribution of each species to the pressure:

$$p_i = p\frac{T_i}{T_i + T_e}, \qquad p_e = p\frac{T_e}{T_i + T_e}.$$

For conservation of energy we need $\mathcal{E} = \mathcal{E}_i + \mathcal{E}_e$. The constitutive relations

$$\mathcal{E}_i = \frac{\alpha_i}{2}p_i + \rho_i u_i^2/2, \tag{7.1}$$

$$\mathcal{E}_e = \frac{\alpha_e}{2}p_e + \rho_e u_e^2/2, \tag{7.2}$$

where α_s is the number of energy modes of species s ($\alpha_s = 3$ for purely translational modes, e.g. for electrons and protons), add to give the net constitutive relation

$$\mathcal{E} = \frac{\alpha_{\mathrm{MHD}}}{2}p + \rho u^2/2 + \tilde{m}_i \tilde{m}_e \frac{J^2}{2\rho},$$

where α_{MHD}, the effective number of degrees of freedom of the MHD system, is determined by the requirements that

$$p_{\mathrm{MHD}} = p_i + p_e \qquad \text{and} \qquad \alpha_{\mathrm{MHD}} p_{\mathrm{MHD}} = \alpha_i p_i + \alpha_e p_e.$$

Dividing these equations, we can deduce that the effective number of degrees of freedom of the MHD system is the average of the number of degrees of freedom of each component, weighted according to the ratio of pressures (or thermal energy densities, or temperatures in the quasineutral case):

$$\alpha_{\mathrm{MHD}} = \frac{\alpha_i p_i + \alpha_e p_e}{p_i + p_e} \approx \frac{\alpha_i T_i + \alpha_e T_e}{T_i + T_e}$$

In summary, the more correct MHD constitutive relation

$$\begin{aligned}
\mathcal{E} &= \frac{\alpha_{\mathrm{MHD}}}{2} p + \rho u^2/2 + \tilde{m}_i \tilde{m}_e \frac{J^2}{2\rho}, \\
\alpha_{\mathrm{MHD}} &\approx \frac{\alpha_i T_i + \alpha_e T_e}{T_i + T_e}
\end{aligned}$$

fails to agree with the constitutive relation that we use for MHD,

$$\mathcal{E} = \frac{\alpha_{\mathrm{MHD}}}{2} p_{\mathrm{MHD}} + \rho u^2/2,$$

unless we incorporate the diffusion kinetic energy into the pressure:

$$p_{\mathrm{MHD}} := p + \frac{2}{\alpha_{\mathrm{MHD}}} \tilde{m}_i \tilde{m}_e \frac{J^2}{2\rho}.$$

We use the following procedure to obtain the energies and pressures:

1. Compute the kinetic energy of each species and add to get the total species kinetic energy. If this exceeds the total MHD kinetic energy, then it will not be possible to satisfy energy conservation without modifying the diffusion velocities we computed (on the assumption of the pre-Maxwell Ampere's law and quasineutrality).

   $$\mathcal{E}^{\mathrm{k}}_{\mathrm{species}} := (1/2)[\rho_i u_i^2 + \rho_e u_e^2]; \qquad \mathcal{E}_{\mathrm{MHD}} = (\alpha_{\mathrm{MHD}}/2) p_{\mathrm{MHD}} + \rho u^2/2.$$

2. Subtract the total species kinetic energy from the total gas energy to get the total species thermal energy.

   $$\mathcal{E}^{\mathrm{t}}_{\mathrm{species}} = \mathcal{E}_{\mathrm{MHD}} - \mathcal{E}^{\mathrm{k}}_{\mathrm{species}}$$

3. Split the total species thermal energy using the ratio of temperatures (and, more generally, of the gas constants, if they differ) of the two species.

   $$\begin{aligned}
   \mathcal{E}^{\mathrm{t}}_i &= \frac{\alpha_i T_i}{\alpha_i T_i + \alpha_e T_e} \mathcal{E}^{\mathrm{t}}_{\mathrm{species}}; &\qquad p_i &= (2/\alpha_i)\mathcal{E}^{\mathrm{t}}_i; \\
   \mathcal{E}^{\mathrm{t}}_e &= \frac{\alpha_e T_e}{\alpha_i T_i + \alpha_e T_e} \mathcal{E}^{\mathrm{t}}_{\mathrm{species}}; &\qquad p_e &= (2/\alpha_e)\mathcal{E}^{\mathrm{t}}_e.
   \end{aligned}$$

Finally, the electric field is supplied by Ohm's law (6.2):

$$\mathbf{E} = \eta \mathbf{J} + \mathbf{B} \times \mathbf{u} + \frac{\tilde{m}_i - \tilde{m}_e}{\rho} \mathbf{J} \times \mathbf{B} + \frac{1}{\rho} \nabla \cdot (\tilde{m}_e \mathbb{P}_i - \tilde{m}_i \mathbb{P}_e) + \frac{\tilde{m}_i \tilde{m}_e}{\rho} \Big(\partial_t \mathbf{J} + \nabla \cdot \big(\mathbf{u}\mathbf{J} + \mathbf{J}\mathbf{u} + \frac{\tilde{m}_e - \tilde{m}_i}{\rho} \mathbf{J}\mathbf{J} \big) \Big).$$