Review: division, factoring, and root-finding.

(This is a brush-up from last semester.)

The division algorithm allows us to write any two polynomials $f(x)$ and $p(x) \neq 0$ uniquely in the form $f(x) = p(x) \cdot q(x) + r(x)$, where $q(x)$ is called the *quotient* and $r(x)$ is called the *remainder*.

In lecture we used the division algorithm to show that factoring is the same as root-finding. In particular, the remainder theorem tells you that $f(c)$ is the remainder when you divide $f(x)$ by $(x - c)$. (To see this, write $f(x) = (x - c) \cdot q(x) + r$. Then $f(c) = r$.)

Problems.

1. **Constructing a polynomial with given roots.**

   Construct a polynomial with real coefficients that has exactly the given zeros and degree.

   (a) $3 - 4i$, degree 2
   (b) $3 - 4i, 7$, degree 3
   (c) $3 - 4i, 7$, degree 4

2. **Rational roots.**

   Completely factor the following polynomials. HINT: The rational roots theorem and synthetic division can be a big help.

   (a) $x^3 - x^2 + 4x - 4$
   (b) $2x^3 - 5x^2 - 11x - 4$

3. **Complex arithmetic.**

   Let $z_1 = 2\,\mathrm{cis}\,(150°)$ and $z_2 = 3\,\mathrm{cis}\,(-60°)$. Find:

   (a) z_1 and z_2 in $a + bi$ form
   (b) z_2/z_1
   (c) $z_2 - z_1$
   (d) z_2^5
   (e) All solutions z to $z^2 = z_1$.

4. **Derivative of a polynomial with a root of multiplicity greater than 1.**

   Have each person in your group invent a polynomial p that has a root of multiplicity 2. Now differentiate it to get the polynomial p'. Factor p'. Do you notice a common pattern? Do you think that this happens in general? Can you prove it?

5. **Factoring a quadratic by depressing it.**

   Suppose you want to factor a quadratic polynomial, i.e. a polynomial of the form $Ax^2 + Bx + C$. If you divide by A, you will get an equation of the form $p = x^2 + \beta x + \gamma$. We wish to put this equation in the "depressed" form $y^2 + c$. To accomplish this, substitute $x = y - \alpha$ and expand out to get an equation of the form $y^2 + by + c$. What should α be in order to make the y-term zero? Choose this α, solve for x, and expand your solution in terms of the original parameters of the quadratic polynomial (A,B, and C). Do you get a familiar formula?

6. **Factoring cubics part I: depressing a cubic.**

   Ever wonder how to get a general formula for the factors of a cubic polynomial? Begin with a general cubic, $Ax^3 + Bx^2 + Cx + D$. Divide by A to get a cubic of the form $x^3 + \beta x^2 + \gamma x + \delta$. We wish to put this equation in the "depressed" form $y^3 + cy + d$. The trick is to make a substitution of the form $x = y - \alpha$. What should α be in order to make the y^2 term zero?

7. **Factoring cubics part II: solving a depressed cubic.**

   Suppose you are trying to factor a depressed cubic, $p = y^3 + cy + d$. So you are trying to solve the equation $y^3 + cy + d = 0$. The trick is to look for a solution of the form $y = s - t$. Plug this in. Show that we've got a solution if:

   $$c = t^3 - s^3, \quad d = 3st$$

   So if we can solve this system for s and t in terms of c and d, we're done. Do you see how to solve it?

8. **Factoring cubics part III: try an example!**

   Now let's do an example. To have a nice example to test out our method, let's make up a cubic where we already know the roots. I don't know, how about $p(x) = (x-3)(x^2+1)$? Or maybe you would prefer real roots, e.g. $p(x) = (x - 5)(x + 1)(x + 4)$? Multiply out the factored polynomial that you pick and then use the methods in the previous two sections to factor it. Does it work?

   You also might try starting with a random depressed cubic, such as $p(x) = x^3 - 3x + 1$. Compute the roots. Then find the zeros by graphing or using Newton's method. Do your answers agree?

9. **Factoring a quartic.**

To factor a quartic equation, first figure out a substitution that "depresses" it (i.e. gets rid of the cube term). Then assume that you can write it as a product of quadratic polynomials with generic coefficients. Multiply the quadratics to get a quartic and equate the coefficients with the coefficients of the depressed quartic. This gives a system of equations which, with some strategic elimination of variables, you can reduce to the problem of finding the roots of a cubic. Voilà!

10. **Factoring a quintic.**

    Show that it is impossible to represent the roots of the polynomial $f(x) = x^5 - x - 1$ using addition, subtraction, multiplication, division, or extraction of roots. Note that this result shows that it is impossible to find a general formula for the roots of a polynomial of degree 5 or higher. (Hint: read up on Galois theory! :)