X-point analysis

by E. Alec Johnson, Dec 2010

We can gain insight into the mechanisms for reconnection of magnetic field lines in plasma by considering configurations highly constrained by symmetries.

We here refer to an "out-of-plane" (z) axis as an "X-point" if (1) plasma quantities are invariant under translations along the out-of-plane axis (so the equations are independent of the out-of-plane coordinate and we can regard the out-of-plane axis as a point) and (2) on the out-of-plane axis the magnetic field is parallel to the axis.

A study of magnetic reconnection in the vicinity of an X-point should give general insight into magnetic reconnection in regions where the magnetic field is nonvanishing or where there is a magnetic null line (since the symmetries approximate local conditions). Another possibility — reconnection near a magnetic null point (where the magnetic field vanishes at an isolated point) — is a distinct case which requires independent study.

We additionally often impose rotational or reflectional symmetries: symmetry under 180 degree rotation around the z-axis or symmetry under reflection across a pair of orthogonal planes through the z-axis. Symmetry across a plane implies the absence of a guide field, because the magnetic field is a pseudovector, which means it is negated upon reflection. Symmetry across a pair of orthogonal planes through the z axis implies symmetry under 180 degree rotation around the z-axis since reflecting across two orthogonal planes effects a 180 degree rotation. I remark that if the magnetic field is a linear function of space then symmetry across a plane containing the z-axis also implies symmetry across the orthogonal plane through the z-axis (consider eigenvectors).

1 Basic equations

Faraday's law asserts that

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$

where $\mathbf{B}$ is magnetic field and $\mathbf{E}$ is electric field.

If there is a velocity field $\mathbf{v}$ and a ϕ (e.g. 0) for which $\mathbf{E} = \mathbf{B} \times \mathbf{v} + \nabla\phi$, then Faraday's law becomes

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{v}) = 0,$$

which asserts that the magnetic flux is convected by $\mathbf{v}$; the magnetic field lines are thus frozen in the plasma and the topology of the magnetic field lines cannot change. Such a velocity field $\mathbf{v}$ is called a **flux-transporting flow**.

In a plasma each species must satisfy the momentum equation,

$$\rho_s d_t \mathbf{u}_s + \nabla \cdot \mathbb{P}_s = (q_s/m_s)\rho_s(\mathbf{E}_s + \mathbf{u}_s \times \mathbf{B}_s) + \mathbf{R}_s,$$

where $\mathbf{R}$ is resistive drag due to collisions with other species, $\mathbb{P}$ is the pressure tensor, $\mathbf{u}$ is species bulk velocity, ρ is species mass density, and q/m is charge-to-mass ratio.

If the inertia $d_t\mathbf{u}$, the pressure term $\nabla \cdot \mathbb{P}$, and the resistive drag $\mathbf{R}$ are all neglible, then $\mathbf{E} = \mathbf{B} \times \mathbf{u}$ and magnetic flux cannot slip through the species.

2 X-point configuration and relationships

Suppose all quantities are independent of the "out-of-plane" axis z.

2.1 Out-of-plane electric field gives rate of magnetic reconnection.

The out-of-plane electric field reveals the rate of change of the flux across a curve between any two points. Indeed, taking this curve (without loss of generality) to be a segment of the y-axis from the origin to a point y_1 and invoking Faraday's law,

$$\begin{aligned} d_t(\text{Flux})(t) &= \int_0^{y_1} \partial_t B_1 \, dy = -\int_0^{y_1} \partial_y E_3 \, dy \\ &= E_3(0) - E_3(y_1). \end{aligned}$$

Suppose that there is symmetry under 180-degree about the z-axis. Then on the z-axis all vectors must be parallel to the z-axis. This contrains magnetic reconnection.

Suppose $\mathbf{v}$ is a flux-transporting flow. If there is an anchor point in the domain (e.g. infinity or a conducting wall) where the out-of-plane electric field is zero and the flux-transporting flow is constant, then the flux across the line between the anchor point and the x-point would have to be constant and $E_3(0)$ would have to be zero. We generally regard $E_3(0)$ as the rate of reconnection at the x-point.

Since all vectors must be out-of-plane at the origin, the momentum equation reduces to its out-of-plane component, the $\mathbf{u} \times \mathbf{B}$ term disappears, and the material derivative simplifies to a partial derivative:

$$\rho_s \partial_t \mathbf{u}_s + \nabla \cdot \mathbb{P}_s = (q_s/m_s)\rho_s \mathbf{E}_s + \mathbf{R}_s.$$

So the out-of-plane component of the electric field can be nonzero only if the resistive drag, the divergence of the pressure, or the inertial term is nonzero.

2.2 Steady collisionless reconnection needs agyrotropy.

I claim that nonsingular steady reconnection in collisionless plasma requires that $\nabla \cdot \mathbb{P}_s$ be nonzero at the origin. So suppose that the resistivity $\mathbf{R}_s$ and the inertial term $\rho_s \partial_t \mathbf{u}_s$ are both zero and that the reconnection electric field $\mathbf{E}_s$ is nonzero. If $\nabla \cdot \mathbb{P}_s = 0$ at the X-point, then $\rho_s = 0$ at the X-point, which is a singularity.

For $\nabla \cdot \mathbb{P}_s$ to be nonzero at the X-point, the pressure cannot be isotropic in the vicinity of the origin. Otherwise, $\nabla \cdot \mathbb{P}_s = \nabla \cdot (p_s \mathbb{I}) = \nabla p_s$, which must be out-of-plane (and hence zero) at the origin.

More generally, the pressure cannot be gyrotropic in the vicinity of the origin. Suppose otherwise. Then $\mathbb{P}_s = p_\parallel \mathbf{bb} + p_\perp(\mathbb{I} - \mathbf{bb}) = \mathbf{bb}(p_\parallel - p_\perp) + p_\perp \mathbb{I}$, where $\mathbf{b} := \frac{\mathbf{B}}{|\mathbf{B}|}$. So, using that $\nabla \cdot \mathbf{B} = 0$, $\nabla \cdot \mathbb{P}_s = \nabla p_\perp + \mathbf{B} \cdot \nabla \left(\frac{\mathbf{B}}{\mathbf{B} \cdot \mathbf{B}} (p_\parallel - p_\perp) \right)$, which must be zero at the x-point since rotational symmetry implies that ∇ and $\mathbf{B} \cdot \nabla$ must both be zero at the x-point.

This proof has a (singularity) "hole" in it if $\mathbf{B}$ vanishes at the origin.

Suppose there is reflectional symmetry across the x-z plane and across the y-z plane (so $\mathbf{B} = 0$ at the x-point). Then the magnetic field on the y-axis must be parallel to the x-axis and the magnetic field on the x-axis must be parallel to the y-axis. Thus, gyrotropy implies that along the x-axis $P_{xx} = P_{zz}$ and $P_{xz} = 0$ and along the y-axis $P_{yy} = P_{zz}$ and $P_{yz} = 0$. So at the x-point $(\nabla \cdot \mathbb{P})_3 = \partial_x P_{xz} + \partial_y P_{yz} = 0$. Note also that in this case gyrotropy would imply isotropy at the X-point (where $\mathbf{B}$ vanishes). In fact, gyrotropy generically implies isotropy at an isolated null point except e.g. in the case of antiparallel magnetic field lines, where the linearization of the magnetic field has a nilpotent matrix.

References

[Vasyliunas75] Vytenis M. Vasyliunas, *Theoretical Models of Magnetic Field Line Merging, 1*, Reviews of Geophysics and Space Physics, Vol. 13, No. 1, 1975.

[HeKuBi04] Michael Hesse, Masha Kuznetsova, and Joachim Birn, *The role of electron heat flux in guide-field magnetic reconnection*, Physics of Plasmas, Vol. 11, No. 12, 2004.