Use of the Yee scheme to enforce the E-field divergence constraint in Maxwell's equations.

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1 Problem.

We are developing a finite-volume method to solve the 2-fluid ideal model of plasma. This plasma model is a forced hyperbolic system, which we can write as a conservation law with state variables $\underline{q}$, flux $\underline{\underline{f}}(\underline{q})$, and source term $\underline{s}(\underline{q})$:

$$\partial_t \underline{q} + \underline{\nabla} \cdot \underline{\underline{f}}(\underline{q}) = \underline{s}(\underline{q}).$$

In this system the equation for the electric field is the completed Ampere law,

$$\partial_t \mathbf{E} - c\nabla \times (c\mathbf{B}) = -\frac{1}{\epsilon_0}\mathbf{J}.$$

We want a numerical solver for this system that maintains the physical constraint

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\sigma.$$

2 One-dimensional problem.

In the one-dimensional case, partial derivatives with respect to the second and third coordinates are taken to be zero. So the constraint equation becomes

$$E^1_x = \frac{1}{\epsilon}\sigma,$$

and the equation that governs the evolution of E^1 is simply

$$\partial_t E^1 = -\frac{1}{\epsilon}J^1. \tag{2.1}$$

3 Numerical scheme.

In one dimension, the Yee scheme says that we stagger the first component of the electric field with respect to the other quantities and attempt to maintain the discrete divergence condition

$$\frac{E^1_{i+1/2} - E^1_{i-1/2}}{\Delta x} = \frac{\sigma_i}{\epsilon}. \tag{3.1}$$

We will require that our update be of the form

$$E^{1,n+1}_{i+1/2} = E^{1,n}_{i+1/2} - \frac{\Delta t}{\epsilon} J^{1,n+1/2}_{i+1/2}, \tag{3.2}$$

where $J^{1,n+1/2}_{i+1/2}$ is supposed to be a second-order estimate of the current on the right side of (2.1). We ascertain the constraints on this quantity by taking the discrete divergence of the update equation (3.2):

$$\frac{E^{1,n+1}_{i+1/2} - E^{1,n+1}_{i-1/2}}{\Delta x} = \frac{E^{1,n}_{i+1/2} - E^{1,n}_{i-1/2}}{\Delta x} - \frac{\Delta t}{\epsilon}\frac{J^{1,n+1/2}_{i+1/2} - J^{1,n+1/2}_{i-1/2}}{\Delta x}. \tag{3.3}$$

Invoking requirement (3.1), this simply becomes

$$\sigma^{n+1}_i = \sigma^n_i - \frac{\Delta t}{\Delta x}[J^{1,n+1/2}_{i+1/2} - J^{1,n+1/2}_{i-1/2}]. \tag{3.4}$$

Since the net charge density is conserved (i.e. has no source term), this just says that the values $J^{1,n+1/2}_{i+1/2}$ are the numerical fluxes of the charge density.

3.1 Algorithm

Here is the outline of the algorithm to advance one time step:

1. Average the staggered values of E^1 to obtain cell-centered values:

$$E^1_i = \frac{E^1_{i+1/2} + E^1_{i-1/2}}{2}$$

2. Use wave propagation to update all state values. (We will discard the updated values of E^1_i that this produces.) This computation involves computing values of numerical flux for the state variables, including the charge densities. Retain the computations of flux for the charge densities of the ions and electrons and add them to get a net charge flux:

$$J^{1,n+1/2}_{i+1/2} = J^{1,n+1/2}_{\text{ion},i+1/2} + J^{1,n+1/2}_{\text{electron},i+1/2}$$

3. Update the first component of the electric field by means of (3.2):

$$E^{1,n+1}_{i+1/2} = E^{1,n}_{i+1/2} - \frac{\Delta t}{\epsilon} J^{1,n+1/2}_{i+1/2}.$$