Gaussian-moment two-fluid MHD relaxation closure for sustained collisionless fast magnetic reconnection

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[This is a slides version of the poster I presented.]

Abstract

We propose a Gaussian-BGK relaxation closure for the heat flux (and viscosity) for Gaussian-moment two-fluid MHD. We argue that this is the simplest fluid model that can be expected to resolve the pressure tensor near the X-point for fast antiparallel magnetic reconnection: two-fluid effects are needed for collisionless fast reconnection, extended moments are needed to resolve the strong agyrotropy that arises in the pressure tensor near the X-point, and nonzero viscosity and heat flux are necessary to admit sustained reconnection without developing a temperature singularity near the X-point.

Background: two-fluid models

The starting point for deriving two-species plasma models is the kinetic-Maxwell system, which evolves the particle densities $f_s(t, \mathbf{x}, \mathbf{v})$ and the electromagnetic field $(\mathbf{B}, \mathbf{E})$. The standard model of gas dynamics is the **Maxwellian-moment** (5-moment) model, which evolves the 5 physically conserved moments of the kinetic equation. The **Gaussian-moment** (10-moment) model instead evolves all 10 quadratic monomial moments.

Kinetic-Maxwell system

- **Kinetic equations:**

$$\partial_t f_i + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_i + \mathbf{a}_i \cdot \nabla_{\mathbf{v}} f_i = C_i + C_{ie}$$

$$\partial_t f_e + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_e + \mathbf{a}_e \cdot \nabla_{\mathbf{v}} f_e = C_e + C_{ei}$$

- **Lorentz force law**

$$\mathbf{a}_i = \frac{q_i}{m_i} (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathbf{a}_e = \frac{q_e}{m_e} (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- **Maxwell's equations:**

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$$

$$\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = \mathbf{J}/\epsilon_0$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = \sigma/\epsilon_0$$

$$\sigma = \sum_s \frac{q_s}{m_s} \int f_s \, d\mathbf{v}$$

$$\mathbf{J} = \sum_s \frac{q_s}{m_s} \int \mathbf{v} f_s \, d\mathbf{v}$$

Gaussian(10)-moment model:

- **moments:**

$$\begin{bmatrix} \rho_s \\ \rho_s \mathbf{u}_s \\ \mathbb{P}_s \end{bmatrix} = \int \begin{bmatrix} 1 \\ \mathbf{v} \\ \mathbf{cc} \end{bmatrix} f_s \, d\mathbf{v}$$

$$\mathbf{c}_s := \mathbf{v} - \mathbf{u}_s$$

- **closure:**

$$\mathbb{R}_s = \int \mathbf{c}_s \mathbf{c}_s \, C_s \, d\mathbf{v}$$

$$\begin{bmatrix} \mathbf{R}_s \\ \mathbb{Q}_s \end{bmatrix} = \int \begin{bmatrix} \mathbf{v} \\ \mathbf{c}_s \mathbf{c}_s \end{bmatrix} C_{sp} \, d\mathbf{v}$$

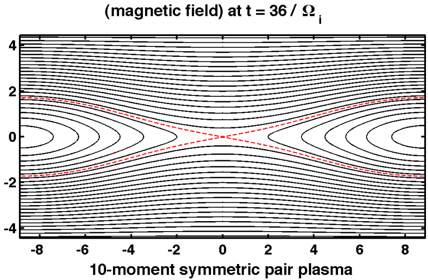
$$\underline{\underline{q}}_s = \int \mathbf{c}_s \mathbf{c}_s \mathbf{c}_s \, f_s \, d\mathbf{c}_s$$

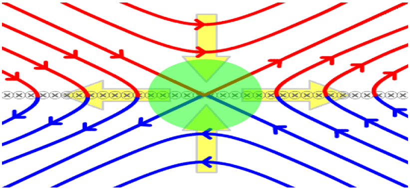
Maxwell(5)-moment model:

$$p_s = \tfrac{1}{3} \operatorname{tr} \mathbb{P}_s, \quad Q_s = \tfrac{1}{2} \operatorname{tr} \mathbb{Q}_s, \quad \mathbf{q}_s = \tfrac{1}{2} \operatorname{tr} \underline{\underline{q}}_s.$$

MHD models assume quasineutrality ($\sigma \approx 0$) and neglect the displacement current $\partial_t \mathbf{E}$ and can be derived assuming the limit $\mathbf{c} \to \infty$. MHD models thus evolve a single density evolution equation and a single momentum evolution equation. Two-fluid MHD evolves separate energy evolution equations for each species.

Define a ***symmetric 2D*** problem to be a 2D problem symmetric under 180-degree rotation about the origin (0). In our simulations of symmetric 2D reconnection the origin is an X-point of the magnetic field:





This first half of the poster identifies requirements for fast magnetic reconnection by analyzing the solution near the X-point. We argue that, for accurate resolution of the electron pressure tensor near the X-point, a fluid model of fast reconnection (1) must resolve two-fluid effects, (2) should resolve strong pressure anisotropy, and (3) must admit viscosity and heat flow.

All equations in part A assume a steady-state solution to a symmetric 2D problem and are evaluated at the origin (0).

1. Ohm's law: fast reconnection needs two-fluid effects.

Ohm's law is net electrical current evolution solved for the electric field. Assuming symmetry across the X-point, the steady-state Ohm's law evaluated at the X-point reads

$$\mathbf{E}^{\parallel} = (\boldsymbol{\eta} \cdot \mathbf{J})^{\parallel} + \frac{1}{e\rho} \left[\nabla \cdot (m_{\mathrm{e}}\mathbb{P}_{\mathrm{i}} - m_{\mathrm{i}}\mathbb{P}_{\mathrm{e}})\right]^{\parallel} \qquad \text{at 0 for } \partial_t = 0.$$

Fast reconnection is nearly collisionless, so the resistive term $\boldsymbol{\eta} \cdot \mathbf{J}$ should be negligible.

For *pair plasma*, the pressure term is zero unless the pressure tensors of the two species are allowed to differ. In fact, kinetic simulations of collisionless antiparallel reconnection admit fast rates of reconnection [BeBh07], and we get similar rates using a two-fluid Gaussian-moment model of pair plasma with pressure isotropization [Jo11].

For *hydrogen plasma*, the electron pressure term chiefly supports reconnection, and the Hall term $\frac{m_{\mathrm{i}} - m_{\mathrm{e}}}{e\rho} \mathbf{J} \times \mathbf{B}$, although zero at the X-point, appears to accelerate the rate of reconnection [ShDrRoDe01].

2. Pressure anisotropy at X-point needs an extended-moment model.

For antiparallel reconnection, the pressure tensor becomes strongly agyrotropic in the immediate vicinity of the X-point [Br11, ScGr06]. Stress closures for the Maxwellian-moment model assume that the pressure tensor is nearly isotropic. In contrast, the assumptions of the Gaussian-moment model (that the distribution of particle velocities is nearly Gaussian) can hold even for strongly anisotropic pressure. In practice, we have found good agreement of the Gaussian-moment two-fluid model with kinetic simulations [Jo11, JoRo10]:

- Reconnection rates are approximately correct.
- Reconnection is primarily supported by pressure agyrotropy.
- There is qualitatively good resolution of the electron pressure tensor near the X-point even when the pressure becomes strongly agyrotropic.

3. Theory: steady collisionless reconnection requires viscosity & heat flux

For a symmetric 2D problem, the origin is a stagnation point. Informally, we show that steady reconnection is not possible without heat production near the stagnation point and that a mechanism for heat flow is therefore necessary to prevent a heating singularity at the stagnation point. Formally, define a solution to be **nonsingular** if density and pressure are finite, strictly positive, and smooth; we show that a steady-state solution to a symmetric 2D problem must be singular if viscosity or heat flux is absent.

3a. Steady collisionless reconnection requires viscosity.

By *Faraday's law* the rate of reconnection is $\mathbf{E}^{\parallel}(0)$ (the out-of-plane electric field evaluated at the origin). Momentum evolution implies

$$\mathbf{E}^{\parallel}(0) = \frac{-\mathbf{R}_{\mathrm{s}}^{\parallel}}{\sigma_{\mathrm{s}}} + \frac{(\nabla \cdot \mathbb{P}_{\mathrm{s}})^{\parallel}}{\sigma_{\mathrm{s}}} \qquad \text{at 0 for } \partial_t = 0, \tag{1}$$

where σ_{s} is charge density. For collisionless reconnection the drag force $\mathbf{R}_{\mathrm{s}}$ should be negligible. If the pressure is isotropric or gyrotropic in a neighborhood of 0, then $\nabla \cdot \mathbb{P}_{\mathrm{s}}$ is zero. That is, inviscid models do not admit steady reconnection [HeKuBi04].

Viscous models generate heat near the X-point. Symmetry implies that the X-point is a stagnation point. An adiabatic fluid model provides no mechanism for heat to dissipate away from the X-point. As a result, viscous adiabatic models develop a temperature singularity near the X-point when used to simulate sustained reconnection. Numerically, when we simulated the GEM magnetic reconnection challenge problem using an adiabatic Gaussian-moment model with pressure isotropization (viscosity), shortly after the peak reconnection rate temperature singularities developed near the X-point. Theoretically, we have the following steady-state result:

Theorem [Jo11]. *For a 2D problem invariant under 180-degree rotation about 0 (the origin), steady-state nonsingular magnetic reconnection is impossible without heat flux for a Maxwellian-moment or Gaussian-moment model that uses linear (gyrotropic) closure relations that satisfy a positive-definiteness condition and respect entropy (in the Maxwellian limit).*

Proof (Maxwellian case)

Let $'$ denote a partial derivative (∂_x or ∂_y) evaluated at 0. Conservation of mass and pressure evolution imply the **entropy evolution equation**:

$$p_s \mathbf{u}_s \cdot \nabla s = 2\boldsymbol{e}_s^\circ : \boldsymbol{\mu}_s : \boldsymbol{e}_s^\circ - \nabla \cdot \mathbf{q}_s + Q_s, \tag{2}$$

where $\boldsymbol{e}_s^\circ$ is deviatoric strain, $-\mathbb{P}_s^\circ = 2\boldsymbol{\mu}_s : \boldsymbol{e}_s^\circ$ is deviatoric stress, and $\boldsymbol{\mu}_s$ is the viscosity tensor. Assume that $\mathbf{q}_s = 0$ near 0. Evaluating equation (2) at 0 and invoking symmetries yields $\boldsymbol{e}_s^\circ : \boldsymbol{\mu}_s : \boldsymbol{e}_s^\circ = -Q_s$. Assume that $\boldsymbol{\mu}$ is positive-definite. Assume that thermal heat exchange conserves energy: $Q_i + Q_e = 0$. So Q_s must be zero, so $\boldsymbol{e}_s^\circ = 0$ at 0. Evaluating the second derivative of equation (2) at 0 and invoking symmetries yields $(\boldsymbol{e}_s^\circ)' : \boldsymbol{\mu} : (\boldsymbol{e}_s^\circ)' = -Q_s''$, which by conservation of energy ($Q_i'' + Q_e'' = 0$) must be nonpositive for one of the two species (which we take to be s) for differentiation along two orthogonal directions. Using that $\boldsymbol{\mu}$ is positive-definite, $(\boldsymbol{e}_s^\circ)' = 0$. Therefore, $-(\mathbb{P}_s^\circ)' = 2(\boldsymbol{\mu}_s : \boldsymbol{e}_s^\circ)' = 0$. Since this relation holds for two orthogonal directions, $\nabla \mathbb{P}_s = 0$ at 0, so $\nabla \cdot \mathbb{P}_s = 0$ at 0. So equation (1) says that $\mathbf{E}^{\|}(0) = 0$, i.e., there is no reconnection. □

A similar proof can be given for the Gaussian case by differentiating the Gaussian-moment entropy evolution equation.

Proof (Gaussian case)

Let $'$ denote a partial derivative (∂_x or ∂_y) evaluated at 0. Conservation of mass and pressure evolution imply the **entropy evolution equation**:

$$n_s \mathbf{u}_s \cdot \nabla s = -2\tau^{-1}\mathbb{P}_s^{-1} : \mathbf{C} : \mathbb{P}_s^\circ - \mathbb{P}_s^{-1} : \nabla \cdot \mathbf{q}_s + \mathbb{P}_s^{-1} : \mathbb{Q}_s, \qquad (3)$$

where $\mathbb{R}_s := \tau^{-1}\mathbf{C} : \mathbb{P}^\circ$ is traceless. Assume that $\underline{\underline{\underline{q}}}_s = 0$ near 0. Evaluating equation (3) at 0 and invoking symmetries yields

$$0 = -2\tau^{-1}(\mathbb{P}_s^{-1}) : \mathbf{C} : (\mathbb{P}_s^\circ) + \mathbb{P}_s^{-1} : \mathbb{Q}_s. \qquad (4)$$

Assume that $\mathbf{C}$ satisfies the positive-definiteness criterion $-(\mathbb{P}_s^{-1}) : \mathbf{C} : (\mathbb{P}_s^\circ) \geq 0$. Assume that a linear closure is used for $\mathbb{Q}_i$ and $\mathbb{Q}_e$ (thermal heat exchange) in terms of $\mathbb{P}_i$ and $\mathbb{P}_e$ which respects total gas-dynamic entropy at 0. Then $\mathbb{P}_s^\circ = 0$ at 0. Evaluating the second derivative of equation (3) at 0 and invoking symmetries yields

$$0 = -2\tau^{-1}(\mathbb{P}_s^{-1})' : \mathbf{C} : (\mathbb{P}_s^\circ)' + (\mathbb{P}_s^{-1} : \mathbb{Q}_s)''. \qquad (5)$$

Using that $\mathbf{C}$ is positive-definite, $(\mathbb{P}_s^\circ)' = 0$ for a species s. That is, $\nabla \mathbb{P}_s = 0$ at 0, so $\nabla \cdot \mathbb{P}_s = 0$ at 0. □

In this second half we present, as the simplest model satisfying these requirements, a Gaussian-BGK closure of Gaussian-moment two-fluid MHD. A Gaussian-BGK collision operator relaxes the particle velocity distribution toward a Gaussian distribution. We assume a Gaussian-BGK collision operator and use a Chapman-Enskog expansion to derive a closure for Maxwellian-moment and Gaussian-moment MHD.

Equations of (Maxwellian-moment) two-fluid MHD

Magnetic field:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0$$

Ohm's law:

$$\mathbf{E} = \boldsymbol{\eta} \cdot \mathbf{J} + \mathbf{B} \times \mathbf{u} + \frac{m_i - m_e}{e\rho} \mathbf{J} \times \mathbf{B} + \frac{1}{e\rho} \nabla \cdot (m_e \mathbb{P}_i - m_i \mathbb{P}_e) + \frac{m_i m_e}{e^2 \rho} \left[\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{uJ} + \mathbf{Ju} - \frac{m_i - m_e}{e\rho} \mathbf{JJ} \right) \right]$$

Mass and momentum:

$$\partial_t \rho + \nabla \cdot (\mathbf{u}\rho) = 0$$

$$\rho d_t \mathbf{u} + \nabla \cdot (\mathbb{P}_i + \mathbb{P}_e + \mathbb{P}^d) = \mathbf{J} \times \mathbf{B}$$

Pressure evolution:

$$\frac{3}{2} n d_t T_i + p_i \nabla \cdot \mathbf{u}_i + \mathbb{P}_i^\circ : \nabla \mathbf{u}_i + \nabla \cdot \mathbf{q}_i = Q_i$$

$$\frac{3}{2} n d_t T_e + p_e \nabla \cdot \mathbf{u}_e + \mathbb{P}_e^\circ : \nabla \mathbf{u}_e + \nabla \cdot \mathbf{q}_e = Q_e$$

Closures:

$$\mathbb{P}_s^\circ = -2 \boldsymbol{\mu}_s : \mathbf{e}_s^\circ$$

$$\mathbf{q}_s = -\mathbf{k}_s \cdot \nabla T_s$$

$$(Q_s = Q_s^f + Q_s^t)$$

Definitions:

$$d_t = \partial_t + \mathbf{u}_s \cdot \nabla$$

$$\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B}$$

$$\mathbf{e}_s^\circ = (\nabla \mathbf{u}_s)^\circ$$

$$\rho = (m_i + m_e) n$$

$$p_s = n T_s$$

$$\mathbb{P}_s = p_s \mathbb{I} + \mathbb{P}_s^\circ$$

$$\mathbb{P}^d = \rho_i \mathbf{w}_i \mathbf{w}_i + \rho_e \mathbf{w}_e \mathbf{w}_e$$

$$\mathbf{w}_i = \frac{m_e \mathbf{J}}{e\rho}, \quad \mathbf{w}_e = -\frac{m_i \mathbf{J}}{e\rho}$$

The Gaussian-moment model evolves full pressure tensors rather than scalar pressure; the equations are identical to those of Maxwellian-moment two-fluid MHD except for the following.

Pressure tensor evolution

$$n d_t \mathbb{T}_i + \mathrm{Sym2}(\mathbb{P}_i \cdot \nabla \mathbf{u}_i) + \nabla \cdot \underline{\underline{q}}_i = \frac{q_i}{m_i} \mathrm{Sym2}(\mathbb{P}_i \times \mathbf{B}) + \mathbb{R}_i + \mathbb{Q}_i$$

$$n d_t \mathbb{T}_e + \mathrm{Sym2}(\mathbb{P}_e \cdot \nabla \mathbf{u}_e) + \nabla \cdot \underline{\underline{q}}_e = \frac{q_e}{m_e} \mathrm{Sym2}(\mathbb{P}_e \times \mathbf{B}) + \mathbb{R}_e + \mathbb{Q}_e$$

Closures:

$$\mathbb{R}_s = -\mathbb{P}_s^\circ / \tau_s$$

$$\underline{\underline{q}}_s = -\tfrac{2}{5} \mathbf{K}_s \,\vdots\, \mathrm{Sym3}\,(\pi \cdot \nabla \mathbb{T}_s)$$

$$(\mathbb{Q}_s = \mathbb{Q}_s^f + \mathbb{Q}_s^t)$$

Definitions:

$$\pi = \frac{\mathbb{P}}{p} = \frac{\mathbb{T}}{T}$$

$$\mathrm{Sym2} = X \mapsto X + X^T$$

$$\mathrm{Sym3} = \left\{ \begin{matrix} \text{thrice symmetric part} \\ \text{of third-order tensor} \end{matrix} \right\}$$

Assuming a Gaussian-BGK intraspecies collision operator and performing a Chapman-Enskog expansion about an assumed distribution yields closures for deviatoric pressure and heat flux.

For the Maxwell-moment model we expand about a Maxwellian distribution and obtain implicit closures for heat flux and deviatoric pressure [Woods04]:

$$\mathbf{q} + \widetilde{\varpi}\mathbf{b} \times \mathbf{q} = -k\nabla T, \qquad (6)$$

$$\mathbb{P}^\circ + \mathrm{Sym2}(\varpi\mathbf{b} \times \mathbb{P}^\circ) = -\mu 2\boldsymbol{e}^\circ, \qquad (7)$$

where μ is viscosity, k is heat conductivity, $\varpi := \tau\omega_c$ is the gyrofrequency per momentum diffusion rate, $\widetilde{\varpi} := \varpi/\Pr$ is the gyrofrequency per thermal diffusion rate, and Pr is the *Prandtl number;* the gyrofrequency is $\omega_c := q|\mathbf{B}|/m$, and $\mathbf{b} := \mathbf{B}/|\mathbf{B}|$.

For the Gaussian-moment model we expand about a Gaussian distribution and obtain the relaxation closure $\mathbb{R}_\mathrm{s} = -\mathbb{P}^\circ_\mathrm{s}/\tau_\mathrm{s}$ and an implicit closure relation for the heat flux tensor [Jo11, McGr08]:

$$\boxed{\underline{\underline{\underline{q}}} + \mathrm{Sym3}(\widetilde{\varpi}\mathbf{b} \times \underline{\underline{\underline{q}}}) = -\tfrac{2}{5}k\,\mathrm{Sym3}\,(\boldsymbol{\pi} \cdot \nabla\mathbb{T})}. \qquad (8)$$

Explicit intraspecies closure (viscosity and heat flux)

In this frame the species index s *is suppressed. All products of tensors are* ***splice symmetric products*** *satisfying* $2(AB)_{j_1 j_2 k_1 k_2} := A_{j_1 k_1} B_{j_2 k_2} + B_{j_1 k_1} A_{j_2 k_2}$ *and*

$$
\begin{aligned}
&3!(ABC)_{j_1 j_2 j_3 k_1 k_2 k_3} \\
&:= A_{j_1 k_1} B_{j_2 k_2} C_{j_3 k_3} + \quad A_{j_1 k_1} C_{j_2 k_2} B_{j_3 k_3} \\
&+ B_{j_1 k_1} A_{j_2 k_2} C_{j_3 k_3} + \quad B_{j_1 k_1} C_{j_2 k_2} A_{j_3 k_3} \\
&+ C_{j_1 k_1} A_{j_2 k_2} B_{j_3 k_3} + \quad C_{j_1 k_1} B_{j_2 k_2} A_{j_3 k_3}
\end{aligned}
$$

(so permute the letters and leave the indices unchanged).

Definitions:

$$
\begin{aligned}
\delta_\parallel &:= \mathbf{bb}, \\
\delta_\perp &:= \mathbb{I} - \mathbf{bb}, \\
\delta_\wedge &:= \mathbf{b} \times \mathbb{I}.
\end{aligned}
$$

Solving equations (6–7) for $\mathbf{q}$ and $\mathbb{P}^\circ$ gives

$$
\begin{aligned}
\mathbf{q} &= -k\widetilde{\mathbf{k}} \cdot \nabla T, \\
\mathbb{P}^\circ &= -2\mu \widetilde{\boldsymbol{\mu}} : \boldsymbol{e}^\circ,
\end{aligned}
$$

where [Woods04]

$$
\begin{aligned}
\widetilde{\mathbf{k}} &= \delta_\parallel + \tfrac{1}{1+\widetilde{\varpi}^2}(\delta_\perp - \widetilde{\varpi}\delta_\wedge), \\
\widetilde{\boldsymbol{\mu}} &= \tfrac{1}{2}(3\delta_\parallel^2 + \delta_\perp^2) + \tfrac{2}{1+\varpi^2}(\delta_\perp \delta_\parallel - \varpi \delta_\wedge \delta_\parallel) \\
&\quad + \tfrac{1}{1+4\varpi^2}\left(\tfrac{1}{2}(\delta_\perp^2 - \delta_\wedge^2) - 2\varpi\delta_\wedge\delta_\perp\right).
\end{aligned}
$$

Solving equation (8) for $\underline{\underline{\underline{q}}}$ gives [Jo11]

$$
\begin{aligned}
\underline{\underline{\underline{q}}} &= -\tfrac{2}{5}k\widetilde{\mathbf{K}} \,\vdots\, \mathrm{Sym3}(\boldsymbol{\pi} \cdot \nabla \mathbb{T}), \\
\widetilde{\mathbf{K}} &= \left(\delta_\parallel^3 + \tfrac{3}{2}\delta_\parallel(\delta_\perp^2 + \delta_\wedge^2)\right) \\
&\quad + \tfrac{3}{1+\widetilde{\varpi}^2}\left(\delta_\perp \delta_\parallel^2 - \widetilde{\varpi}\delta_\wedge\delta_\parallel^2\right) \\
&\quad + \tfrac{3}{1+4\widetilde{\varpi}^2}\left(\tfrac{1}{2}(\delta_\perp^2 - \delta_\wedge^2)\delta_\parallel - 2\widetilde{\varpi}\delta_\wedge\delta_\perp\delta_\parallel\right) \\
&\quad + (k_0\delta_\perp^3 + k_1\delta_\wedge\delta_\perp^2 + k_2\delta_\wedge^2\delta_\perp + k_3\delta_\wedge^3),
\end{aligned}
$$

where

$$
\begin{aligned}
k_3 &:= \frac{-6\widetilde{\varpi}^3}{1 + 10\widetilde{\varpi}^2 + 9\widetilde{\varpi}^4} &&= -(2/3)\widetilde{\varpi}^{-1} + \mathcal{O}(\widetilde{\varpi}^{-3}), \\
k_2 &:= \frac{6\widetilde{\varpi}^2 + 3\widetilde{\varpi}(1 + 3\widetilde{\varpi}^2)k_3}{1 + 7\widetilde{\varpi}^2} &&= \mathcal{O}(\widetilde{\varpi}^{-2}), \\
k_1 &:= \frac{-3\widetilde{\varpi} + 2\widetilde{\varpi}k_2}{1 + 3\widetilde{\varpi}^2} &&= -\widetilde{\varpi}^{-1} + \mathcal{O}(\widetilde{\varpi}^{-3}), \\
k_0 &:= 1 + \widetilde{\varpi}k_1 &&= \mathcal{O}(\widetilde{\varpi}^{-2}).
\end{aligned}
$$

For computational efficiency instead use **splice products**,

$$
\begin{aligned}
(AB)'_{j_1 j_2 k_1 k_2} &:= A_{j_1 k_1} B_{j_2 k_2}, \\
(ABC)'_{j_1 j_2 j_3 k_1 k_2 k_3} &:= A_{j_1 k_1} B_{j_2 k_2} C_{j_3 k_3},
\end{aligned}
$$

and symmetrize at the end, e.g.

$$
\underline{\underline{\underline{q}}}_{\mathrm{s}} = -\tfrac{2}{5}k_{\mathrm{s}}\,\mathrm{Sym}\left(\widetilde{\mathbf{K}}'_{\mathrm{s}} \,\vdots\, \mathrm{Sym3}\,(\boldsymbol{\pi} \cdot \nabla \mathbb{T}_{\mathrm{s}})\right).
$$

Interspecies closure (friction and thermal equilibration)

For collisionless reconnection the interspecies collisional terms should not be necessary for fast reconnection and should be small in comparison to the intraspecies collisional terms. Nevertheless, for completeness we give a linear relaxation closure.

For **thermal equilibration** one can relax toward the average temperature

$$Q^{\mathrm{t}}_{\mathrm{s}} = \tfrac{3}{2} K\, n^2 (T_0 - T_s),$$

where $2T_0 := T_i + T_e$, or toward an average temperature tensor

$$\mathbb{Q}^{\mathrm{t}}_{\mathrm{s}} = K\, n^2 (\mathbb{T}_0 - \mathbb{T}_s),$$

where $2\mathbb{T}_0 := \widetilde{\mathbb{T}}_i + \widetilde{\mathbb{T}}_e$ and

$$\widetilde{\mathbb{T}}_{\mathrm{s}} := \nu' T_{\mathrm{s}} \mathbb{I} + \nu \mathbb{T}_{\mathrm{s}},$$

where $\nu' + \nu = 1$, $0 \le \nu' \le \frac{3}{2}$ and ν' might be 1 or Pr^{-1}. Note that the equilibration rate is nK.

Frictional heating can be allocated among species in inverse proportion to particle mass:

$$Q^{\mathrm{f}} := Q^{\mathrm{f}}_{\mathrm{i}} + Q^{\mathrm{f}}_{\mathrm{e}} = \boldsymbol{\eta} : \mathbf{JJ}$$

$$m_{\mathrm{i}} Q^{\mathrm{f}}_{\mathrm{i}} = m_{\mathrm{e}} Q^{\mathrm{f}}_{\mathrm{e}}$$

The frictional tensor heating also must be allocated among directions:

$$\mathbb{Q}^{\mathrm{f}} = (\alpha_{\parallel} - \alpha_{\perp})\, \mathrm{Sym2}(\boldsymbol{\eta} \cdot \mathbf{JJ}) + \alpha_{\perp} \boldsymbol{\eta} : \mathbf{JJ}\, \mathbb{I},$$

$$\mathbb{Q}^{\mathrm{f}}_{\mathrm{i}} = \frac{m_{\mathrm{e}}}{m_{\mathrm{e}} + m_{\mathrm{i}}} \mathbb{Q}^{\mathrm{f}},$$

$$\mathbb{Q}^{\mathrm{f}}_{\mathrm{e}} = \frac{m_{\mathrm{i}}}{m_{\mathrm{e}} + m_{\mathrm{i}}} \mathbb{Q}^{\mathrm{f}}.$$

where $\alpha_{\parallel} + 2\alpha_{\perp} = 1$ and $0 \le \alpha_{\parallel} \le 1$.

Relaxation coefficients

Diffusion

$$\mu_s = \tau_s n T_s$$
$$\tfrac{2}{5} k_s = \frac{\mu_s}{m_s \,\mathrm{Pr}_s}$$

Relaxation periods

$$\tau_0 := \frac{12\pi^{3/2}}{\ln \Lambda}\left(\frac{\epsilon_0}{e^2}\right)^2$$
$$n\tau'_{ss} := \tau_0 \sqrt{m_s \det(\mathbb{T}_s)}$$

Braginskii

$$\tau_i^{\mathrm{Br}} := \tau'_{ii}$$
$$\tau_e^{\mathrm{Br}} := \tfrac{1}{\sqrt{2}}\tau'_{ee}$$
$$\tau_i = .96\tau'_{ii}$$
$$\tau_e = .52\tau'_{ee}$$
$$\mathrm{Pr}_i = .61 \approx \tfrac{2}{3}$$
$$\mathrm{Pr}_e = .58 \approx \tfrac{2}{3}$$

Note that we define the relaxation periods in terms of $\sqrt{\det(\mathbb{T}_s)}$ rather than $T^{3/2}$ in order to prevent the closure for the heat flux tensor from violating positivity.

Neglectable (interspecies)

$$K^{-1} := \tau_0 \frac{m_i m_e}{\sqrt{2}}\left(\frac{T_i}{m_i} + \frac{T_e}{m_e}\right)^{3/2}$$
$$2\tau_{ei}^{\epsilon,\mathrm{Br}} = (K\,n)^{-1} \approx \tau_e^{\mathrm{Br}}\frac{m_i}{m_e}$$
$$\eta_0 := \frac{m_e}{e^2 n \tau_e^{\mathrm{Br}}}, \qquad \eta_\parallel := .51\eta_0, \qquad \lim_{\varpi\to\infty}\eta_\perp = \eta_0$$

Braginskii's closures are based on Coulomb collisions. In collisionless systems, relaxation is not really mediated by Coulomb collisions, and interspecies relaxation terms should be smaller than this.

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