What can the Implicit Moment Method do?

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Abstract: iPic3D implements the Implicit Moment Method (IMM). IMM is a semi-implicit solver for the classical kinetic-Maxwell system. IMM is implicit in electromagnetic field and current but not in current flux. IMM can thus step over oscillations and light waves (but must resolve electron sounds waves). IMM's explicit current flux makes updated current a linear function of updated electric field, making the field update independent of the particle update.

Classical 2-fluid MHD assumes instantaneous light waves ($c \to \infty$). Classical one-fluid MHD also assumes instantaneous oscillations ($e \to \infty$). So IMM is the natural method to smoothly and efficiently handle the classical two-fluid and one-fluid MHD limits for fixed ion-electron mass ratio.

For truly relativistic problems, the appropriate semi-implicit method is not IMM but the Implicit Source method, because relativistic one-fluid MHD assumes instantaneous source term oscillations ($e \to \infty$) but has no fast waves ($c \nrightarrow \infty$).

IMM is a general electromagnetic solver and can be used independent of whether kinetics are modeled with particles, the Vlasov equation, or a fluid model.

Outline

particle evolution:

$$d_t\mathbf{x}_\mathrm{p} = \mathbf{v}_\mathrm{p},$$

$$d_t\mathbf{u}_\mathrm{p} = e\frac{q_p^\#}{m_p}\left(\mathbf{v}_p \times \mathbf{B}(\mathbf{x}_p) + \mathbf{E}(\mathbf{x}_p)\right) + \mathbf{r}_\mathrm{p},$$

$$\gamma_\mathrm{p}^2 := 1 + (\mathbf{u}_\mathrm{p}/c)^2,$$

$$\mathbf{v}_\mathrm{p} := \mathbf{u}_\mathrm{p}/\gamma_\mathrm{p}.$$

electromagnetic field:

$$\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0,$$

$$-c^{-2}\partial_t\mathbf{E} + \nabla\times\mathbf{B} = \mu_0\mathbf{J},$$

$$\nabla\cdot\mathbf{B} = 0, \quad c^{-2}\nabla\cdot\mathbf{E} = \mu_0\sigma,$$

charge density and current:

$$\sigma(\mathbf{x}) := e\textstyle\sum_p S_p(\mathbf{x} - \mathbf{x}_p)q_p^\#,$$

$$\mathbf{J}(\mathbf{x}) := e\textstyle\sum_p S_p(\mathbf{x} - \mathbf{x}_p)q_p^\#\mathbf{v}_p.$$

To abbreviate, drop the particle summation index p and the independent variable **x** and write $q := eq_p^\#$:

$$\sigma := \textstyle\sum qS \quad \text{(charge)},$$

$$\rho := \textstyle\sum mS \quad \text{(mass)},$$

$$\mathbf{J} := \textstyle\sum \mathbf{v}qS \quad \text{(current)},$$

$$\mathbf{M} := \textstyle\sum \mathbf{u}mS \quad \text{(momentum)},$$

$$\mathcal{E} := \textstyle\sum \frac{1}{2}|\mathbf{v}|^2 mS \quad \text{(energy)}.$$

Collisional acceleration $\mathbf{r}_\mathrm{p}$ must conserve momentum and energy:

$$\sum \mathbf{r}_\mathrm{p} m_\mathrm{p} S_\mathrm{p} = 0 \quad \text{(momentum)},$$

$$\sum \mathbf{r}_\mathrm{p}\cdot\mathbf{v}_\mathrm{p} m_\mathrm{p} S_\mathrm{p} = 0 \quad \text{(energy)}.$$

Shape motion:

$$\boxed{\partial_t S = -\mathbf{v}\cdot\nabla S}, \qquad (1)$$

where we have used the chain rule.

Lorentz force.

$$\boxed{\dot{\mathbf{u}} = \frac{q}{m}\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)}$$

Energy change.

$$\boxed{\dot\gamma = \mathbf{v}\cdot\dot{\mathbf{u}} = \frac{q}{m}\mathbf{v}\cdot\mathbf{E}}, \qquad (2)$$

because $\gamma^2 = 1 + (\mathbf{u}/c)^2$, so $\gamma\dot\gamma = \mathbf{u}\cdot\dot{\mathbf{u}}/c^2$.

Velocity change.

$$\boxed{\dot{\mathbf{v}} = \frac{q}{\gamma m}\left(\mathbf{E} - \frac{\mathbf{vv}}{c^2}\cdot\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)},$$

which follows from differentiating $\mathbf{u} = \gamma\mathbf{v}$, to get $\dot{\mathbf{u}} = \dot\gamma\mathbf{v} + \gamma\dot{\mathbf{v}}$, i.e. $\gamma\dot{\mathbf{v}} = \dot{\mathbf{u}} - \mathbf{vv}\cdot\dot{\mathbf{u}}$.

General moment evolution.

Moments definitions are of the form $\sum\chi S$. To derive fluid equations, differentiate the moment definition:

$$\partial_t\sum\chi S = \sum\chi\partial_t S + \sum\dot\chi S$$

and use the basic derivatives to the left. Using $\nabla\mathbf{v} = 0$ gives:

$$\boxed{\begin{aligned}&\partial_t\sum\chi S + \nabla\cdot\sum\mathbf{v}\chi S = \sum\dot\chi S,\\ &\dot\chi = \frac{\partial\chi}{\partial\mathbf{u}}\cdot\dot{\mathbf{u}} = \frac{\partial\chi}{\partial\mathbf{v}}\cdot\dot{\mathbf{v}},\end{aligned}}$$

Momentum density ($\chi = m\mathbf{u}$).

$$\begin{aligned}&\partial_t\textstyle\sum_{\mathrm{p}} m_{\mathrm{p}}\mathbf{u}_{\mathrm{p}}S(\mathbf{x}-\mathbf{x}_{\mathrm{p}})\\ &\quad+\nabla\cdot\textstyle\sum_{\mathrm{p}}\mathbf{v}_{\mathrm{p}}\mathbf{u}_{\mathrm{p}}m_{\mathrm{p}}S(\mathbf{x}-\mathbf{x}_{\mathrm{p}})\\ &=\left(\textstyle\sum q_{\mathrm{p}}S(\mathbf{x}-\mathbf{x}_{\mathrm{p}})\right)\mathbf{E}\\ &\quad+\left(\textstyle\sum q_{\mathrm{p}}\mathbf{v}_{\mathrm{p}}S(\mathbf{x}-\mathbf{x}_{\mathrm{p}})\right)\times\mathbf{B}.\end{aligned}$$

Relativistic case.

Mass density ($\chi = m$).

$$\partial_t \rho + \nabla \cdot \sum \mathbf{v} m S = 0$$

Momentum density ($\chi = m\mathbf{u}$).

$$\partial_t \mathbf{M} + \nabla \cdot \sum \mathbf{v}\mathbf{u} m S = \sigma \mathbf{E} + \mathbf{J} \times \mathbf{B}$$

Energy density ($\chi = mc^2\gamma$).

$$\partial_t \mathcal{E} + \nabla \cdot \mathbf{M} = \mathbf{J} \cdot \mathbf{E}$$

Charge density ($\chi = q$)

$$\partial_t \sigma + \nabla \cdot \mathbf{J} = 0$$

Current density ($\chi = q\mathbf{v}$)

$$\boxed{\partial_t \mathbf{J} + \nabla \cdot \mathcal{P} = \sum S \frac{q^2}{\gamma m} \left(\mathbb{I} - \frac{\mathbf{v}\mathbf{v}}{c^2} \right) \cdot \mathbf{E} + \sum S \frac{q^2 \mathbf{v}}{\gamma m} \times \mathbf{B}},$$

where $\mathcal{P} := \sum q\mathbf{v}\mathbf{v}$.

Classical case.

Mass density ($\chi = m$)

$$\partial_t \rho + \nabla \cdot \mathbf{M} = 0$$

Momentum density ($\chi = m\mathbf{v}$)

$$\partial_t \mathbf{M} + \nabla \cdot \sum \mathbf{v}\mathbf{v} m S = \sigma \mathbf{E} + \mathbf{J} \times \mathbf{B}$$

Energy density ($\chi = \frac{1}{2} m |\mathbf{v}|^2$)

$$\partial_t \mathcal{E} + \nabla \cdot \sum \tfrac{1}{2} \mathbf{v} v^2 m S = \mathbf{J} \cdot \mathbf{E}$$

Charge density ($\chi = q$).

$$\partial_t \sigma + \nabla \cdot \mathbf{J} = 0$$

Current density ($\chi = q\mathbf{v}$) **for species** s

$$\boxed{\partial_t \mathbf{J}_\mathrm{s} + \nabla \cdot \mathcal{P}_\mathrm{s} = \frac{q_\mathrm{s}}{m_\mathrm{s}} (\sigma_\mathrm{s} \mathbf{E} + \mathbf{J}_\mathrm{s} \times \mathbf{B})}$$

where $\mathcal{P}_\mathrm{s}$ restricts to species s.

Outline

1. Model equations
2. Limit models
3. Semi-implicit schemes
4. The Implicit Moment Method (IMM)

numerics framework for model limits

Goal: numerics that efficiently handles model limits.

- Want: PIC code that works in the MHD limit.
- Program: study deriviation of physics equations and discretize consistently.

Limit models

Classical ideal MHD assumes three limits:

process	limit	consequence
collisions	$\nu_{ss} \to \infty$	2-fluid Maxwell
light waves	$c \to \infty$	2-fluid MHD
oscillations	$e \to \infty$	1-fluid MHD

Subsets of these three limits yield a commuting cube of $8 = 2^3$ **limit models**.

Use SI for classical limit:

- SI-looking units arise naturally from a generic nondimensionalization:
  - e is gyrofrequency,
  - c is c/v_t.
- In Gaussian units, **B** has incorrect scaling.
  - Trying to treat spatial (**B**) and temporal (**E**) components of field tensor symmetrically. . .
  - but trying to treat space and time asymmetrically ($c \neq 1$).
- Therefore, in Gaussian units:
  - cannot analyze classical limit $c \to \infty$.
  - cannot handle classical limit in asympototic preserving manner.

Fluid model limits ($\nu_{ss} = \infty$)

$$\begin{array}{ccc}
\text{2fluid-Maxwell} & \xrightarrow[\text{IMM}]{c\to\infty} & \text{2fluid MHD} \\
e\to\infty \downarrow \text{ImpSrc} & & \downarrow e\to\infty \\
\text{“relativistic” MHD} & \xrightarrow{c\to\infty} & \text{classical MHD}
\end{array}$$

Kinetic model limits ($\nu_{ss} = 0$)

$$\begin{array}{ccc}
\text{Kinetic-Maxwell} & \xrightarrow[\text{IMM}]{c\to\infty} & \text{Kinetic-XMHD} \\
e\to\infty \downarrow \text{ImpSrc} & & \downarrow e\to\infty \\
\text{Kinetic}_{\parallel}\text{MHD} & \xrightarrow{c\to\infty} & \text{Classical Kinetic}_{\parallel}\text{MHD}
\end{array}$$

Other limits are possible:

- The limit $m_i/m_e \to \infty$ for fixed e yields Hall MHD.
- I assume m_i/m_e is fixed (finite).

1. **kinetic-Maxwell**

   $\downarrow$ $\nu_{\rm ss} \to \infty$ (fast collisions)

2. **two-fluid Maxwell**: one gas for each species: $\rho_{\rm s}(\mathbf{x})$, $\mathbf{V}_{\rm s}(\mathbf{x})$, $\mathcal{E}_{\rm s}(\mathbf{x})$

   $\downarrow$ $c \to \infty$ (fast light waves)

3. **two-fluid MHD**: $\mathbf{E} = \mathbf{V} \times \mathbf{B} + \eta\mathbf{J} + \cdots$

   $\downarrow$ $e \to \infty$ (fast plasma oscillations)

4. **classical MHD**: $\mathbf{E} = \mathbf{V} \times \mathbf{B} + \eta\mathbf{J}$

gas evolution:

$$\partial_t \rho_s + \nabla \cdot (\mathbf{V}_s \rho_s) = 0,$$
$$\rho_s d_t^s \mathbf{V}_s + \nabla p_s + \nabla \cdot \mathbb{P}_s^\circ = \sigma_s \mathbf{E} + \mathbf{J}_s \times \mathbf{B} + \mathbf{R}_s$$
$$\rho_s d_t^s \mathcal{E}_s + p_s \nabla \cdot \mathbf{V}_s + \mathbb{P}_s^\circ : \nabla \mathbf{V}_s + \nabla \cdot \mathbf{q}_s = Q_s$$

electromagnetic field:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$-c^{-2} \partial_t \mathbf{E} + \nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$
$$\nabla \cdot \mathbf{B} = 0, \quad c^{-2} \nabla \cdot \mathbf{E} = \mu_0 \sigma.$$
$$\sigma := \sigma_i + \sigma_e, \quad \sigma_s := \frac{e q_s^\#}{m_s} \rho_s.$$
$$\mathbf{J} := \mathbf{J}_i + \mathbf{J}_e, \quad \mathbf{J}_s := \sigma_s \mathbf{V}_s.$$

Closures:

$$\left[\begin{aligned}
\frac{\mathbf{R}_e}{en} &\approx \boldsymbol{\eta} \cdot \mathbf{J} + \boldsymbol{\beta}_e \cdot \mathbf{q}_e, \\
\mathbf{R}_i &= -\mathbf{R}_e, \\
Q_s &=: Q_s^{ex} + Q_s^{fr}, \\
Q_s^{ex} &\approx \tfrac{3}{2} K_s\, n^2 (T_0 - T_s), \\
Q^{fr} &:= Q_i^{fr} + Q_e^{fr} \\
&\approx \boldsymbol{\eta} : \mathbf{J}\mathbf{J} + \boldsymbol{\beta}_e : \mathbf{q}_e \mathbf{J}, \\
Q_i^{fr} &= Q_e^{fr} m_e / m_i, \\
\mathbb{P}_s^\circ &\approx -2 \boldsymbol{\mu}_s : \nabla \mathbf{V}_s^\circ, \\
\mathbf{q}_s &\approx -\mathbf{k}_s \cdot \nabla T_s.
\end{aligned}\right]$$

Definitions:

$$d_t^s := \partial_t + \mathbf{V}_s \cdot \nabla,$$
$$\mathbf{c}_s := \mathbf{v} - \mathbf{V}_s,$$
$$n_s := \rho_s / m_s,$$
$$\mathbb{X}^\circ := \frac{\mathbb{X} + \mathbb{X}^T}{2} - \frac{\mathbb{I}\, \mathrm{tr}\, \mathbb{X}}{3}.$$

mass and momentum (total):

$$\partial_t \rho + \nabla \cdot (\mathbf{V}\rho) = 0$$

$$\rho d_t \mathbf{V} + \nabla \cdot (\mathbb{P}_\mathrm{i} + \mathbb{P}_\mathrm{e} + \mathbb{P}^\mathrm{d}) = \mathbf{J} \times \mathbf{B}$$

energy evolution (per species):

$$\rho_\mathrm{i} d_t \mathcal{E}_\mathrm{i} + p_\mathrm{i} \nabla \cdot \mathbf{V}_\mathrm{i} + \mathbb{P}^\circ_\mathrm{i} : \nabla \mathbf{V}_\mathrm{i} + \nabla \cdot \mathbf{q}_\mathrm{i} = Q_\mathrm{i},$$

$$\rho_\mathrm{e} d_t \mathcal{E}_\mathrm{e} + p_\mathrm{e} \nabla \cdot \mathbf{V}_\mathrm{e} + \mathbb{P}^\circ_\mathrm{e} : \nabla \mathbf{V}_\mathrm{e} + \nabla \cdot \mathbf{q}_\mathrm{e} = Q_\mathrm{e};$$

electromagnetism

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

$$\mathbf{J} = {\mu_0}^{-1} \nabla \times \mathbf{B}$$

Ohm's law (evolution of **J** solved for **E**)

$$\mathbf{E} = \boldsymbol{\eta} \cdot \mathbf{J} + \mathbf{B} \times \mathbf{V} + \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{J} \times \mathbf{B}$$
$$+ \frac{1}{e\rho} \nabla \cdot \left(m_\mathrm{e}(p_\mathrm{i}\mathbb{I} + \mathbb{P}^\circ_\mathrm{i}) - m_\mathrm{i}(p_\mathrm{e}\mathbb{I} + \mathbb{P}^\circ_\mathrm{e}) \right)$$
$$+ \frac{m_\mathrm{i} m_\mathrm{e}}{e^2 \rho} \left[\partial_t \mathbf{J} + \nabla \cdot \left(\mathbf{V}\mathbf{J} + \mathbf{J}\mathbf{V} - \frac{m_\mathrm{i} - m_\mathrm{e}}{e\rho} \mathbf{J}\mathbf{J} \right) \right]$$

Closures (simplified):

$$Q := Q_\mathrm{i} + Q_\mathrm{e}$$
$$\approx \boldsymbol{\eta} : \mathbf{J}\mathbf{J}$$
$$Q_\mathrm{s} = \frac{m_\mathrm{red}}{m_\mathrm{s}} Q,$$
$$\mathbb{P}^\circ_\mathrm{s} \approx -2\boldsymbol{\mu}_\mathrm{s} : \nabla \mathbf{V}^\circ_\mathrm{s},$$
$$\mathbf{q}_\mathrm{s} \approx -\mathbf{k}_\mathrm{s} \cdot \nabla T_\mathrm{s}.$$

Definitions:

$$d_t := \partial_t + \mathbf{V} \cdot \nabla,$$
$$\mathbf{w} = \frac{\mathbf{J}}{en},$$
$$\mathbf{w}_\mathrm{i} = \frac{m_\mathrm{red}}{m_\mathrm{i}} \mathbf{w},$$
$$\mathbf{w}_\mathrm{e} = \frac{-m_\mathrm{red}}{m_\mathrm{e}} \mathbf{w},$$
$$\mathbb{P}^\mathrm{d} := m_\mathrm{red} n \mathbf{w}\mathbf{w},$$
$$m_\mathrm{red}^{-1} := m_\mathrm{e}^{-1} + m_\mathrm{i}^{-1}.$$

Relativistic model hierarchy

1. **kinetic-Maxwell**

   $\downarrow$ $\nu_{\mathrm{ss}} \to \infty$ (fast collisions)

2. **two-fluid Maxwell**: Euler gas for each species: ρ_{s}, $\mathbf{V}_{\mathrm{s}}$, p_{s}

   $\downarrow$ $e \to \infty$ (fast oscillations)

3. **relativistic MHD**

   $\downarrow$ $c \to \infty$ (fast light waves)

4. **classical MHD**: $\mathbf{E} = \mathbf{V} \times \mathbf{B} + \eta \mathbf{J}$

Two-fluid Maxwell:

gas evolution:

$$\partial_t \rho_s + \nabla\cdot(\rho_s \mathbf{V}_s) = 0,$$
$$\rho_s d_t^s \mathbf{V}_s + \nabla p_s = \mathbf{J}_s \times \mathbf{B} + \sigma_s \mathbf{E} + \mathbf{R}_s,$$
$$d_t^s p_s + \gamma p_s \nabla\cdot\mathbf{V}_s = \frac{2}{3}\frac{m_{\mathrm{red}}}{m_s} Q$$

electromagnetic field:

$$\partial_t \mathbf{B} + \nabla\times\mathbf{E} = 0,$$
$$-c^{-2}\partial_t \mathbf{E} + \nabla\times\mathbf{B} = \mu_0 \mathbf{J},$$
$$\nabla\cdot\mathbf{B} = 0, \quad c^{-2}\nabla\cdot\mathbf{E} = \mu_0\sigma,$$
$$\mathbf{J} := \mathbf{J}_i + \mathbf{J}_e, \quad \mathbf{J}_s := \sigma_s \mathbf{V}_s,$$
$$\sigma := \sigma_i + \sigma_e, \quad \sigma_s := \pm\frac{e}{m_s}\rho_s.$$

closure:

$$-\mathbf{R}_i = \mathbf{R}_e = e^2 n_e n_i \boldsymbol{\eta}\cdot(\mathbf{V}_i - \mathbf{V}_e)$$
$$\approx en\boldsymbol{\eta}\cdot\mathbf{J},$$
$$Q = -\sum_s \mathbf{R}_s\cdot\mathbf{V}_s \approx \mathbf{J}\cdot\boldsymbol{\eta}\cdot\mathbf{J}.$$

Quasi-relativistic MHD ($e \to \infty$):

gas evolution:

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{V}) = 0 \qquad \text{(mass)},$$
$$\rho \mathrm{d}_t \mathbf{V} + \nabla p = \mathbf{J}\times\mathbf{B} + \sigma\mathbf{E} \qquad \text{(momentum)},$$
$$d_t p + \gamma p \nabla\cdot\mathbf{V} = \frac{2}{3}\mathbf{J}\cdot\boldsymbol{\eta}\cdot\mathbf{J} \qquad \text{(thermal energy)}.$$

magnetic field:

$$\partial_t \mathbf{B} + \nabla\times\mathbf{E} = 0 \qquad \text{(magnetic field)},$$
$$\mathbf{E} = \mathbf{B}\times\mathbf{V} + \boldsymbol{\eta}\cdot\mathbf{J} \qquad \text{(Ohm's law)},$$
$$\nabla\cdot\mathbf{B} = 0 \qquad \text{(divergence constraint)},$$
$$\mu_0\mathbf{J} := \nabla\times\mathbf{B} - c^{-2}\partial_t\mathbf{E} \qquad \text{(Ampere's law for current)},$$
$$\mu_0\sigma := c^{-2}\nabla\cdot\mathbf{E} \qquad \text{(quasineutrality)}.$$

definitions:

$$d_t^s := \partial_t + \mathbf{V}_s\cdot\nabla, \qquad \gamma := \frac{5}{3},$$
$$d_t := \partial_t + \mathbf{V}\cdot\nabla, \qquad m_{\mathrm{red}}^{-1} := \sum_s m_s^{-1}.$$

Two-fluid Maxwell:

gas evolution:

$$\partial_t \rho_s + \nabla \cdot (\rho_s \mathbf{V}_s) = 0,$$
$$\rho_s d_t^s \mathbf{V}_s + \nabla p_s = \mathbf{J}_s \times \mathbf{B} + \sigma_s \mathbf{E} + \mathbf{R}_s,$$
$$d_t^s p_s + \gamma p_s \nabla \cdot \mathbf{V}_s = \frac{2}{3}\frac{m_{\text{red}}}{m_s} Q$$

electromagnetic field:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$-c^{-2}\partial_t \mathbf{E} + \nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$
$$\nabla \cdot \mathbf{B} = 0, \quad c^{-2}\nabla \cdot \mathbf{E} = \mu_0 \sigma,$$
$$\mathbf{J} := \mathbf{J}_i + \mathbf{J}_e, \quad \mathbf{J}_s := \sigma_s \mathbf{V}_s,$$
$$\sigma := \sigma_i + \sigma_e, \quad \sigma_s := \pm \frac{e}{m_s}\rho_s.$$

closure:

$$-\mathbf{R}_i = \mathbf{R}_e = e^2 n_e n_i \boldsymbol{\eta} \cdot (\mathbf{V}_i - \mathbf{V}_e)$$
$$\approx en\boldsymbol{\eta} \cdot \mathbf{J},$$
$$Q = -\textstyle\sum_s \mathbf{R}_s \cdot \mathbf{V}_s \approx \mathbf{J} \cdot \boldsymbol{\eta} \cdot \mathbf{J}.$$

Classical MHD ($c \to \infty$, $e \to \infty$):

gas evolution:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{V}) = 0 \quad \text{(mass)},$$
$$\rho d_t \mathbf{V} + \nabla p = \mathbf{J} \times \mathbf{B} \quad \text{(momentum)},$$
$$d_t p + \gamma p \nabla \cdot \mathbf{V} = \frac{2}{3}\mathbf{J} \cdot \boldsymbol{\eta} \cdot \mathbf{J} \quad \text{(thermal energy)}.$$

magnetic field:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0 \quad \text{(magnetic field)},$$
$$\mathbf{E} = \mathbf{B} \times \mathbf{V} + \boldsymbol{\eta} \cdot \mathbf{J} \quad \text{(Ohm's law)},$$
$$\nabla \cdot \mathbf{B} = 0 \quad \text{(divergence constraint)},$$
$$\mu_0 \mathbf{J} := \nabla \times \mathbf{B} \quad \text{(Ampere's law for current)},$$
$$\mu_0 \sigma := 0 \quad \text{(neutrality)}.$$

definitions:

$$d_t^s := \partial_t + \mathbf{V}_s \cdot \nabla, \qquad \gamma := \frac{5}{3},$$
$$d_t := \partial_t + \mathbf{V} \cdot \nabla, \qquad m_{\text{red}}^{-1} := \sum_s m_s^{-1}.$$

Outline

Limit models and semi-implicit schemes

Classical ideal MHD assumes that three processes of kinetic-Maxwell are instantaneous:

process	limit	consequence
collisions	$\nu_{ss} \to \infty$	fluid approximation
oscillations	$e \to \infty$	relativistic one-fluid MHD
light waves	$c \to \infty$	classical two-fluid MHD

Explicit schemes must resolve all these processes.

Implicit schemes can step over fast processes, allowing smooth and efficient (asymptotic-preserving) transition to the limit model that assumes the fast process is instantaneous.

Fully implicit schemes can step over all processes including MHD waves but require iterating the entire solve.

Semi-implicit schemes are a cheap alternative that can step only over the needed subset of processes.

Implicit Source (IMEX) schemes step over fast source term processes ($\nu \to \infty$ and $e \to \infty$) but not over fast waves ($c \to \infty$) and are suitable for relativistic codes.

The **Implicit Moment Method (IMM)** steps over all three processes without stepping over any two-fluid MHD waves.

Discretization	must resolve...	AP limit model
Explicit	plasma period [everything]	—
Implicit Source (IMEX)	light waves	relativistic MHD
Implicit Moment (IMM)	electron sound waves	ideal MHD
Fully Implicit	[no restriction]	[implicit MHD]

Start with the basic equations:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = -\mathbf{J}/\epsilon_0,$$
$$\partial_t \mathbf{x}_\mathrm{p} = \mathbf{v}_\mathrm{p},$$
$$\partial_t \mathbf{u}_\mathrm{p} = \frac{q_\mathrm{p}}{m_\mathrm{p}} \left(\mathbf{E}(\mathbf{x}_\mathrm{p}) + \mathbf{v}_\mathrm{p} \times \mathbf{B}(\mathbf{x}_\mathrm{p}) \right).$$

For a second-order discretization, use a leapfrog discretization:

- $\mathbf{B}$ does not need particle velocities to advance, but $\mathbf{E}$ does, so particle velocities $\mathbf{u}_\mathrm{p}$ and current $\mathbf{J}$ should be staggered relative to $\mathbf{E}$.
- $\partial_t X = (X^2 - X^0)/\Delta t$
- $\partial_t Y = (Y^3 - Y^1)/\Delta t$

In the relativistic case, time-split the velocity update for a symplectic method. In full detail:

$$\mathbf{B}^2 = \mathbf{B}^0 - \Delta t \nabla \times \mathbf{E}^1,$$
$$\mathbf{E}^1 = \mathbf{E}^{-1} + \Delta t \left(c^2 \nabla \times \mathbf{B}^0 - \mathbf{J}^0/\epsilon_0 \right),$$
$$\mathbf{u}_\mathrm{p}^* = \mathbf{u}_\mathrm{p}^0 + \frac{q_\mathrm{p} \Delta t}{2 m_\mathrm{p} \gamma_\mathrm{p}^0} (\mathbf{u}_\mathrm{p}^* + \mathbf{u}_\mathrm{p}^0) \times \bar{\mathbf{B}}^1(\mathbf{x}_\mathrm{p}^1),$$
$$\mathbf{u}_\mathrm{p}^2 = \mathbf{u}_\mathrm{p}^* + \Delta t \frac{q_\mathrm{p}}{m_\mathrm{p}} \mathbf{E}^1(\mathbf{x}_\mathrm{p}^1),$$

where $\bar{\mathbf{B}}^1 := \frac{1}{2}\mathbf{B}^0 + \frac{1}{2}\mathbf{B}^2$.

The fully implicit method makes all terms implicit:

$$\partial_t \mathbf{B} + \nabla \times \bar{\mathbf{E}} = 0$$

$$\partial_t \mathbf{E} - c^2 \nabla \times \bar{\mathbf{B}} = -\bar{\mathbf{J}}/\epsilon_0$$

$$\partial_t \mathbf{u}_\mathrm{p} = \frac{q_\mathrm{p}}{m_\mathrm{p}} \left(\bar{\sigma}_\mathrm{p} \bar{\mathbf{E}}(\bar{\mathbf{x}}_p) + \bar{\mathbf{v}}_\mathrm{p} \times \bar{\mathbf{B}}(\bar{\mathbf{x}}_\mathrm{p}) \right),$$

$$\partial_t \mathbf{x}_\mathrm{p} = \bar{\mathbf{v}}_\mathrm{p},$$

$$\partial_t \sigma_\mathrm{s} + \nabla \cdot \bar{\mathbf{J}}_\mathrm{s} = 0,$$

$$\partial_t \mathbf{J}_\mathrm{s} + \nabla \cdot \overline{\mathcal{P}}_\mathrm{s} = \frac{q_\mathrm{s}}{m_\mathrm{s}} \left(\bar{\sigma}_\mathrm{s} \bar{\mathbf{E}} + \bar{\mathbf{J}}_\mathrm{s} \times \bar{\mathbf{B}} \right).$$

Remarks.

- No time step restriction.
- Particle advance must be redone with successive iterations of the field solver.

Use initial values for flux terms. and implicit values in stiff source.

Fields:

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0,$$
$$\partial_t \mathbf{E} - c^2 \nabla \times \mathbf{B} = -\bar{\mathbf{J}}/\epsilon_0.$$

Particles:

$$\partial_t \mathbf{u}_\mathrm{p} = \frac{q_\mathrm{p}}{m_\mathrm{p}} \left(\bar{\mathbf{E}}(\mathbf{x}_\mathrm{p}) + \bar{\mathbf{v}}_\mathrm{p} \times \bar{\mathbf{B}}(\mathbf{x}_\mathrm{p}) \right),$$
$$\partial_t \mathbf{x}_\mathrm{p} = \bar{\mathbf{v}}_\mathrm{p},$$
$$\partial_t \sigma_\mathrm{s} + \nabla \cdot \mathbf{J}_\mathrm{s} = 0.$$

Cassical current:

$$\partial_t \mathbf{J}_\mathrm{s} + \nabla \cdot \mathcal{P}_\mathrm{s} = \frac{q_\mathrm{s}}{m_\mathrm{s}} \left(\bar{\sigma}_\mathrm{s} \bar{\mathbf{E}} + \bar{\mathbf{J}}_\mathrm{s} \times \bar{\mathbf{B}} \right).$$

We designate $n = 0$ as initial time, $n + 1 = 1$ as final time, and time discretization as

$$\partial_t Q \to (Q^1 - Q_0)/\Delta t,$$
$$\nabla F \to F^0,$$
$$\bar{\mathbf{X}} = X^1.$$

Classical case: No source term iteration happens to be needed, because $\bar{\mathbf{v}}$ is linear in $\bar{\mathbf{E}}$. Can sum the response over all particles to eliminate $\bar{\mathbf{J}} = \widehat{\mathbf{J}} + \mathbb{A} \cdot \bar{\mathbf{E}}$ in favor of $\bar{\mathbf{E}}$.

Relativistic case: Must iterate particle velocity advance, but positions need not be advanced, so iterative solve involves no communication between mesh cells.

High-order accuracy: Use an IMEX Runge-Kutta solver.

To linearize the Maxwell source term, modify the fully implicit discretization by replacing the updated charge densities and magnetic field with explicitly evolved values.

$$\partial_t \mathbf{B} + \nabla \times \bar{\mathbf{E}} = 0$$
$$\partial_t \mathbf{E} - c^2 \nabla \times \bar{\mathbf{B}} = -\bar{\mathbf{J}}/\epsilon_0,$$
$$\partial_t \mathbf{v}_\mathrm{p} = \frac{q_\mathrm{p}}{m_\mathrm{p}} \left(\bar{\mathbf{E}}(\bar{\mathbf{x}}'_p) + \bar{\mathbf{v}}_\mathrm{p} \times \bar{\mathbf{B}}'(\bar{\mathbf{x}}'_\mathrm{p}) \right),$$
$$\partial_t \mathbf{x}_\mathrm{p} = \bar{\mathbf{v}}_\mathrm{p},$$
$$\partial_t \sigma_\mathrm{s} + \nabla \cdot \bar{\mathbf{J}}_\mathrm{s} = 0,$$
$$\partial_t \mathbf{J}_\mathrm{s} + \nabla \cdot \mathcal{P}_\mathrm{s} = \frac{q_\mathrm{s}}{m_\mathrm{s}} \left(\sigma'_\mathrm{s} \bar{\mathbf{E}} + \bar{\mathbf{J}}_\mathrm{s} \times \mathbf{B}' \right)$$

For second-order accuracy in relevant terms, use time averages for the implicit terms:

$$\partial_t Q \to (Q^1 - Q^0)/\Delta t,$$
$$\nabla F \to F^0, \bar{\mathbf{X}} = \tfrac{1}{2} X^0 + \tfrac{1}{2} X^1,$$
$$\bar{\mathbf{v}} = \tfrac{1}{2}\mathbf{v}^0 + \tfrac{1}{2}\mathbf{v}^1,$$
$$\bar{\mathbf{J}} = \tfrac{1}{2}\mathbf{J}^0 + \tfrac{1}{2}\mathbf{J}^1,$$
$$\bar{\mathbf{E}} = \tfrac{1}{2}\mathbf{E}^0 + \tfrac{1}{2}\mathbf{E}^1.$$

Observe that in this discretization,

$$\partial_t X = 2(\bar{\mathbf{X}} - X^0)/\Delta t.$$

Divergence constraints. Eliminating **B** from this field discretization and assuming mimetic operators, the field solve is exactly equivalent to

$$c^{-2}\partial_t \mathbf{E} = (\tfrac{1}{2}\Delta t)\left(\nabla^2 \bar{\mathbf{E}} - \nabla(\nabla \cdot \bar{\mathbf{E}})\right) + \left(\nabla \times \mathbf{B}^0 - \mu_0 \bar{\mathbf{J}}\right);$$

following [RicciLapentaBrackbill02], to ensure the divergence constraint error is damped, substitute $(\nabla \cdot \bar{\mathbf{E}}) \to (\mu_0 c^2 \bar{\sigma})$, where $\bar{\sigma} := \sigma^0 - \frac{1}{2}\Delta t \bar{\mathbf{J}}$.

High-order accuracy. Use implicit Euler discretization and plug scheme into Runge-Kutta method.

Classical case. Can eliminate $\bar{\mathbf{J}}$ in favor of $\bar{\mathbf{E}}$ by putting current evolution in the form $\bar{\mathbf{J}} = \widehat{\mathbf{J}} + \mathbb{A} \cdot \bar{\mathbf{E}}$. Defining $\mathbf{B}' = \mathbf{B}^0 - \frac{\Delta t}{2}\nabla \times \mathbf{E}^0$, $\bar{\mathbf{B}}' = \bar{\mathbf{B}}$, $\sigma'_\mathrm{s} = \sigma^0_\mathrm{s} - \frac{\Delta t}{2}\nabla \cdot \mathbf{J}^0_\mathrm{s}$ would yield full second-order accuracy in time. IMM in literature uses $\mathbf{B}' = \bar{\mathbf{B}}' = \mathbf{B}^0$ and $\sigma'_\mathrm{s} = \sigma^0_\mathrm{s}$, Particle advance is solved iteratively, initialized with $\bar{\mathbf{x}}'_p \leftarrow \mathbf{x}^0_p$; two iterations is enough for second-order accuracy.

Relativistic case. Implicit source seems preferable to IMM for the fully relativistic case for multiple reasons:

- Why step over light waves but not over relativistic sound waves or fluid speed? An implicit source time step is much cheaper and involves no long-distance communication.
- The source term system responds nonlinearly to $\bar{\mathbf{E}}$ and is not closed, so an implicit particle velocity advance must be repeated with successive iterations of the field solve.

Outline

Implicit solve

Calculating particle and current advance in response to the electromagnetic field requires solving equations of the form

$$\boxed{\mathbf{U} = \mathbf{V} + \mathbf{U} \times \boldsymbol{\Omega}} \qquad (3)$$

To solve for $\mathbf{U}$, dot and cross both sides with $\boldsymbol{\Omega}$ to get the equations:

$$\mathbf{U} \cdot \boldsymbol{\Omega} = \mathbf{V} \cdot \boldsymbol{\Omega},$$

$$\begin{aligned} \mathbf{U} \times \boldsymbol{\Omega} &= \mathbf{V} \times \boldsymbol{\Omega} + \boldsymbol{\Omega}\boldsymbol{\Omega} \cdot \mathbf{U} - |\boldsymbol{\Omega}|^2 \mathbf{U} \\ &= \mathbf{V} \times \boldsymbol{\Omega} + \boldsymbol{\Omega}\boldsymbol{\Omega} \cdot \mathbf{V} - |\boldsymbol{\Omega}|^2 \mathbf{U}. \end{aligned}$$

So eliminating $\mathbf{U} \times \boldsymbol{\Omega}$ in (3),

$$\mathbf{U}(1 + |\boldsymbol{\Omega}|^2) = (\mathbb{I} - \boldsymbol{\Omega} \times \mathbb{I} + \boldsymbol{\Omega}\boldsymbol{\Omega}) \cdot \mathbf{V}.$$

That is,

$$\boxed{\mathbf{U} = \frac{\mathbb{I} - \boldsymbol{\Omega} \times \mathbb{I} + \boldsymbol{\Omega}\boldsymbol{\Omega}}{1 + |\boldsymbol{\Omega}|^2} \cdot \mathbf{V}}.$$

Equation (3) says that

$$(\mathbb{I} + \boldsymbol{\Omega} \times \mathbb{I}) \cdot \mathbf{U} = \mathbf{V}$$

We thus infer that

$$\boxed{(\mathbb{I} + \boldsymbol{\Omega} \times \mathbb{I})^{-1} = \frac{\mathbb{I} - \boldsymbol{\Omega} \times \mathbb{I} + \boldsymbol{\Omega}\boldsymbol{\Omega}}{1 + |\boldsymbol{\Omega}|^2}}.$$

IMM average current calculation (classical)

Recall classical current evolution:

$$\partial_t \mathbf{J}_s + \nabla\cdot\mathcal{P}_s = \frac{q_s}{m_s}\left(\sigma_s'\bar{\mathbf{E}} + \bar{\mathbf{J}}_s\times\mathbf{B}'\right).$$

Discretize $\partial_t \mathbf{J}_s$ as $\frac{\mathbf{J}_s^1-\mathbf{J}_s^0}{\Delta t} = \frac{\bar{\mathbf{J}}_s-\mathbf{J}_s^0}{\Delta t/2}$. So

$$\bar{\mathbf{J}}_s = \mathbf{J}_s^0 - \frac{\Delta t}{2}\nabla\cdot\mathcal{P}_s + \beta_s(\sigma_s'\bar{\mathbf{E}} + \bar{\mathbf{J}}_s\times\mathbf{B}'),$$

where $\beta_s := \frac{q_s\Delta t}{2m_s}$. This is of the form (3),

$$\mathbf{U} = \mathbf{V} + \mathbf{U}\times\boldsymbol{\Omega},$$

where

$$\mathbf{U} = \bar{\mathbf{J}}_s,$$
$$\mathbf{V} = \mathbf{J}_s^0 - \frac{\Delta t}{2}\nabla\cdot\mathcal{P}_s + \beta_s\sigma_s'\bar{\mathbf{E}},$$
$$\boldsymbol{\Omega} = \beta_s\mathbf{B}.$$

Thus, the linear response of average current to average electric field is given by:

$$\begin{aligned}
\bar{\mathbf{J}} &= \widehat{\mathbf{J}} + \mathbb{A}\cdot\bar{\mathbf{E}},\\
\mathbb{A} &:= \textstyle\sum_s \beta_s\sigma_s'\Pi_s,\\
\widehat{\mathbf{J}} &:= \textstyle\sum_s \widehat{\mathbf{J}}_s,\\
\widehat{\mathbf{J}}_s &:= \Pi_s\cdot\left(\mathbf{J}_s^0 - \frac{\Delta t}{2}\nabla\cdot\mathcal{P}_s\right),\\
\Pi_s &:= \frac{\mathbb{I} - \boldsymbol{\Omega}_s\times\mathbb{I} + \boldsymbol{\Omega}_s\boldsymbol{\Omega}_s}{1+|\boldsymbol{\Omega}_s|^2},\\
\boldsymbol{\Omega}_s &:= \beta_s\mathbf{B}',\\
\beta_s &:= \frac{q_s\Delta t}{2m_s},\\
\sigma_s^0 &:= \textstyle\sum_{p\in s} S(\mathbf{x}-\mathbf{x}_p^0)q_p,\\
\mathbf{J}_s^0 &:= \textstyle\sum_{p\in s} S(\mathbf{x}-\mathbf{x}_p^0)q_p\mathbf{v}_p^0,\\
\mathcal{P}_s &:= \textstyle\sum_{p\in s} S(\mathbf{x}-\mathbf{x}_p^0)q_p\mathbf{v}_p^0\mathbf{v}_p^0.
\end{aligned}$$

The implicit moment method differences Maxwell's evolution equations implicitly as:

$$\nabla \times \bar{\mathbf{E}} + \frac{\bar{\mathbf{B}} - \mathbf{B}^0}{\theta \Delta t} = 0,$$

$$\nabla \times \bar{\mathbf{B}} - \frac{\bar{\mathbf{E}} - \mathbf{E}^0}{c^2 \theta \Delta t} = \mu_0 \bar{\mathbf{J}};$$

where $\bar{\mathbf{X}} := \theta \mathbf{X}^1 + (1-\theta)\mathbf{X}^0$, so $\theta(\mathbf{X}^1 - \mathbf{X}^0) = \bar{\mathbf{X}} - \mathbf{X}^0$. To eliminate $\bar{\mathbf{B}}$ and get an equation implicit in $\bar{\mathbf{E}}$, take the curl of the first equation:

$$(c\theta\Delta t)^2 \nabla \times \nabla \times \bar{\mathbf{E}} + \bar{\mathbf{E}} = \mathbf{E}^0 + c^2\theta\Delta t(\nabla \times \mathbf{B}^0 - \mu_0 \bar{\mathbf{J}}).$$

The implicit moment method assumes that average current responds linearly to average electric field:

$$\bar{\mathbf{J}} = \widehat{\mathbf{J}} + \frac{\chi}{\mu_0 c^2 \theta \Delta t} \cdot \bar{\mathbf{E}},$$

where the "**implicit susceptibility**" tensor χ is defined so as to be unitless.

Substituting for $\bar{\mathbf{J}}$ yields the field equation used in [KamimuraMBLT92]:

$$(c\theta\Delta t)^2 \nabla \times \nabla \times \bar{\mathbf{E}} + (\bar{\mathbf{E}} + \chi \cdot \bar{\mathbf{E}}) = \mathbf{E}^0 + c^2\theta\Delta t(\nabla \times \mathbf{B}^0 - \mu_0 \widehat{\mathbf{J}}). \quad (4)$$

Including the approximate identities

$$\nabla \times \nabla \times \bar{\mathbf{E}} = -\nabla^2 \bar{\mathbf{E}} + \nabla\nabla \cdot \bar{\mathbf{E}},$$

$$\nabla \cdot \bar{\mathbf{E}} = \mu_0 c^2 \bar{\sigma}, \quad (5)$$

where $\bar{\sigma} := \sigma^0 - \theta\Delta t \nabla \cdot \bar{\mathbf{J}}$

gives a numerically overdetermined system; we modify the method to damp diverge error by invoking the discrete divergence contraint (5). With these assumptions,

$$\nabla \cdot \bar{\mathbf{E}} = \mu_0 c^2 \widehat{\sigma} - \nabla \cdot (\chi \cdot \bar{\mathbf{E}}),$$

where $\boxed{\widehat{\sigma} := \sigma^0 - \theta\Delta t \nabla \cdot \widehat{\mathbf{J}}}$.

Substituting for $\nabla \times \nabla \times \bar{\mathbf{E}}$ in equation (4),

$$\boxed{(\bar{\mathbf{E}} + \chi \cdot \bar{\mathbf{E}}) - (c\theta\Delta t)^2 (\nabla^2 \bar{\mathbf{E}} + \nabla\nabla \cdot (\chi \cdot \bar{\mathbf{E}})) = \mathbf{E}^0 + c^2\theta\Delta t(\nabla \times \mathbf{B}^0 - \mu_0(\widehat{\mathbf{J}} + c^2\theta\Delta t \nabla \widehat{\sigma})).}$$

Divergence error is damped by this equation, whereas it neither grows nor decays for (4).

Having found $\bar{\mathbf{E}}$,

$$\mathbf{B}^1 = \mathbf{B}^0 + \Delta t \nabla \times \bar{\mathbf{E}},$$

$$\mathbf{E}^1 = \theta^{-1}\bar{\mathbf{E}} + (1-\theta^{-1})\mathbf{E}^0.$$

Observe that this magnetic field update maintains the condition that the magnetic field is a discrete curl up to machine precision independent of whether mimetic operators are used.

Accurate closure.

This scheme is second-order accurate for $\theta = 1/2$ and first-order accurate for $\frac{1}{2} < \theta \leq 1$; it is unstable for $\theta < 1/2$.

Needed information:

- $\chi := (\mu_0 c^2 \theta \Delta t)\, \mathbb{A}$
- $\widehat{\mathbf{J}}$, σ^0
- $\mathbf{B}^0$, $\mathbf{E}^0$

Implicit Source field solver

The Implicit Source field solver uses exactly the same formula for average current $\bar{\mathbf{J}} = \widehat{\mathbf{J}} + \mathbb{A} \cdot \bar{\mathbf{E}}$ as the Implicit Moment Method, but since the electromagnetic flux is not implicit the update is noniterative.

The Implicit Source system is:

$$\nabla \times \mathbf{E}^0 + \frac{\bar{\mathbf{B}} - \mathbf{B}^0}{\theta \Delta t} = 0,$$

$$\nabla \times \mathbf{B}^0 - \frac{\bar{\mathbf{E}} - \mathbf{E}^0}{c^2 \theta \Delta t} = \mu_0 \left(\widehat{\mathbf{J}} + \frac{\chi}{\mu_0 c^2 \theta \Delta t} \cdot \bar{\mathbf{E}} \right)$$

Solving for $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$ yields the implicit source field update:

$$\boxed{\bar{\mathbf{E}} = (\mathbb{I} + \chi)^{-1} \cdot \left((\nabla \times \mathbf{B}^0 - \mu_0 \widehat{\mathbf{J}}) c^2 \theta \Delta t + \mathbf{E}^0 \right)}$$

$$\bar{\mathbf{B}} = \mathbf{B}^0 + \theta \Delta t \nabla \times \mathbf{E}^0$$

Replacing $\mathbf{B}^0$ and $\mathbf{E}^0$ with a previously computed value of $\bar{\mathbf{B}}$ and $\bar{\mathbf{E}}$ would make this update a black-red Gauss-Seidel update for the Implicit Moment Method that could be used effectively in a multigrid field solver; iterating the particle advance and using consistently updated values for χ and $\widehat{\mathbf{J}}$ would make this a fully implicit field solver.

Accuracy

This scheme is second-order accurate (in time) for $\theta = 1/2$.

Stability

The implicit source method is unable to take a time step in the stability window of the implicit moment method and requires a means of suppressing the finite grid instability.

Divergence constraints

Divergence contraint error in the electric field could be damped using a correction potential, although in fluid simulations I have not found it necessary or helpful to invoke them. Alternatively, charge conservation is possible via moment tracking.

In my experience, maintaining the divergence constraint for the magnetic field is critical; note that this magnetic field update maintains the condition that the magnetic field is a discrete curl up to machine precision independent of whether mimetic operators are used.

Needed information: $\mathbf{B}^0$, $\mathbf{E}^0$, $\widehat{\mathbf{J}}$, and
$\chi := (\mu_0 c^2 \theta \Delta t)\, \mathbb{A}$,

Implicit particle mover (classical)

Particle position and velocity are differenced as

$$\mathbf{x}_\mathrm{p}^1 = \mathbf{x}_\mathrm{p}^0 + \bar{\mathbf{v}}_\mathrm{p}\Delta t,$$
$$\bar{\mathbf{v}}_\mathrm{p} := \tfrac{1}{2}\mathbf{v}_\mathrm{p}^1 + \tfrac{1}{2}\mathbf{v}_\mathrm{p}^0,$$
$$\mathbf{v}_\mathrm{p}^1 = \mathbf{v}_\mathrm{p}^0 + 2\beta_\mathrm{s}\left(\mathbf{E}_\mathrm{p}^\theta + \bar{\mathbf{v}}_\mathrm{p} \times \mathbf{B}_\mathrm{p}^\vartheta\right),$$

where s is the species of particle p, ϑ might equal θ or 0, and $\beta_\mathrm{s} := \frac{q_\mathrm{s}\Delta t}{2m_\mathrm{s}}$. Choosing $\mathbf{B}_\mathrm{p}^\vartheta := \mathbf{B}^\vartheta(\mathbf{x}_\mathrm{p}^0)$ yields an explicit particle advance. Choosing $\mathbf{B}_\mathrm{p}^\vartheta := \mathbf{B}^\vartheta(\bar{\mathbf{x}}_\mathrm{p})$ and $\mathbf{E}_\mathrm{p}^\theta := \mathbf{E}^\theta(\bar{\mathbf{x}}_\mathrm{p})$, where $\bar{\mathbf{x}}_\mathrm{p} := \frac{1}{2}\mathbf{x}_\mathrm{p}^1 + \frac{1}{2}\mathbf{x}_\mathrm{p}^0$, defines an implicit particle advance. Use two iterations beginning with the explicit advance for second-order accuracy.

Eliminating $\mathbf{v}_\mathrm{p}^1$ in favor of $\bar{\mathbf{v}}_\mathrm{p}$,

$$\bar{\mathbf{v}}_\mathrm{p} = \mathbf{v}_\mathrm{p}^0 + \beta_\mathrm{s}\left(\mathbf{E}_\mathrm{p}^\theta + \bar{\mathbf{v}}_\mathrm{p} \times \mathbf{B}_\mathrm{p}^\vartheta\right).$$

This is of the form

$$\mathbf{U} = \mathbf{V} + \mathbf{U} \times \boldsymbol{\Omega},$$

where

$$\mathbf{U} = \bar{\mathbf{v}}_\mathrm{p},$$
$$\mathbf{V} = \mathbf{v}_\mathrm{p}^0 + \beta_\mathrm{s}\mathbf{E}_\mathrm{p}^\theta,$$
$$\boldsymbol{\Omega} = \beta_\mathrm{s}\mathbf{B}_\mathrm{p}^\vartheta.$$

Thus,

$$\bar{\mathbf{v}}_\mathrm{p} = \widehat{\mathbf{v}}_\mathrm{p}^0 + \beta_\mathrm{s}\Pi_\mathrm{p}^\vartheta \cdot \mathbf{E}_\mathrm{p}^\theta,$$
$$\widehat{\mathbf{v}}_\mathrm{p}^0 := \Pi_\mathrm{p}^\vartheta \cdot \mathbf{v},$$
$$\Pi_\mathrm{p}^\vartheta := \frac{\mathbb{I} - \boldsymbol{\Omega}_\mathrm{p} \times \mathbb{I} + \boldsymbol{\Omega}_\mathrm{p}\boldsymbol{\Omega}_\mathrm{p}}{1 + |\boldsymbol{\Omega}_\mathrm{p}|^2},$$
$$\boldsymbol{\Omega}_\mathrm{p} := \beta_\mathrm{s}\mathbf{B}_\mathrm{p}^\vartheta,$$
$$\beta_\mathrm{s} := \frac{q_\mathrm{p}\Delta t}{2m_\mathrm{p}}.$$

Observe that $|\boldsymbol{\Omega}_\mathrm{p}|$ is half the gyrofrequency for particle p. Note that $\vartheta \in \{0, \theta\}$ is chosen based on whether the updated magnetic field is already known. A first-order-accurate field predictor allows for a fully second-order-accurate solve with $\vartheta = \theta = \frac{1}{2}$.

Advance the position and velocity of each particle p via

$$\mathbf{u}^1 = \mathbf{u}^0 + 2\beta_s\left(\mathbf{E}_p^\theta + \bar{\mathbf{v}} \times \mathbf{B}_p^\vartheta\right), \qquad (6)$$

$$\bar{\mathbf{v}} := \bar{\mathbf{u}}/\bar{\gamma},$$

$$\mathbf{x}_p^1 = \mathbf{x}_p^0 + \bar{\mathbf{v}}\Delta t,$$

where $\bar{\mathbf{u}} := \frac{1}{2}\mathbf{u}^1 + \frac{1}{2}\mathbf{u}^0$, $\mathbf{u} = \mathbf{u}_p$ (etc.), $\bar{\gamma} := \frac{1}{2}\gamma^1 + \frac{1}{2}\gamma^0$, and $\beta_s := \frac{q_s\Delta t}{2m_s}$; choosing $\mathbf{B}_p^\vartheta := \mathbf{B}^\vartheta(\mathbf{x}_p^0)$ yields an explicit advance, and choosing $\mathbf{B}_p^\vartheta := \mathbf{B}^\vartheta(\bar{\mathbf{x}}_p)$ and $\mathbf{E}_p^\theta := \mathbf{E}^\theta(\bar{\mathbf{x}}_p)$ defines an implicit advance, where

$$\bar{\mathbf{x}}_p := \frac{1}{2}\mathbf{x}_p^1 + \frac{1}{2}\mathbf{x}_p^0.$$

Use 2 iterations beginning with the explicit advance for second-order accuracy. Since $\mathbf{u}^1 = 2\bar{\mathbf{u}} - \mathbf{u}^0$,

$$\bar{\mathbf{u}} = \mathbf{u}^0 + \beta_s(\mathbf{E}_p^\theta + \bar{\mathbf{v}} \times \bar{\gamma}^{-1}\mathbf{B}_p^\vartheta).$$

This is of the form

$$\mathbf{U} = \mathbf{V} + \mathbf{U} \times \boldsymbol{\Omega}, \qquad (7)$$

where $\mathbf{U} = \bar{\mathbf{u}}$, $\mathbf{V} = \mathbf{u}^0 + \beta_s\mathbf{E}_p^\theta$, and, $\boldsymbol{\Omega} = \bar{\gamma}^{-1}\beta_s\mathbf{B}_p^\vartheta$ is half the gyrofrequency vector. Thus (restoring index p),

$$\boxed{\begin{aligned}
\bar{\mathbf{u}}_p &= \widehat{\mathbf{u}}_p^0 + \beta_s\Pi_p^\vartheta \cdot \mathbf{E}_p^\theta, \\
\widehat{\mathbf{u}}_p^0 &:= \Pi_p^\vartheta \cdot \mathbf{u}_p^0, \\
\Pi_p^\vartheta &:= \frac{\mathbb{I} - \boldsymbol{\Omega}_p \times \mathbb{I} + \boldsymbol{\Omega}_p\boldsymbol{\Omega}_p}{1 + |\boldsymbol{\Omega}_p|^2}, \\
\boldsymbol{\Omega}_p &:= \frac{\beta_p}{\bar{\gamma}_p}\mathbf{B}_p^\vartheta, \\
\beta_s &:= \frac{q_s\Delta t}{2m_s},
\end{aligned}}$$

This system would be an explicit solution if $\bar{\mathbf{x}}_p$ and $\bar{\gamma}_p$ were known.
In the classical case, $\bar{\gamma}_p = 1$.
In the relativistic case, $\gamma^2 = 1 + (\mathbf{u}/c)^2$, so $\bar{\gamma}\mathrm{d}\gamma = \bar{\mathbf{u}} \cdot \mathrm{d}\mathbf{u}/c^2$, i.e., $\mathrm{d}\gamma = \bar{\mathbf{v}} \cdot \mathrm{d}\mathbf{u}/c^2$.

Substituting into (6),

$$\boxed{\bar{\gamma}_p = \gamma_p^0 + \beta_p\mathbf{E}_p^\theta \cdot \bar{\mathbf{v}}'/c^2},$$

where this equality holds if $\bar{\mathbf{v}}' = \bar{\mathbf{v}}_p$. The initialization $\bar{\mathbf{v}}' \leftarrow \mathbf{v}_p^0$ yields an iterative solver. The relativistic implicit moment method is based on this first iteration.

Note that $\vartheta \in \{0, \theta\}$ is chosen based on whether the updated magnetic field is already known. A first-order-accurate field predictor allows for a fully second-order-accurate solve with $\vartheta = \theta = \frac{1}{2}$.

Recall *relativistic current evolution*:

$$\boxed{\partial_t \mathbf{J} + \nabla\cdot\mathcal{P} = \sum S\frac{q^2}{\gamma m}\left(\mathbb{I} - \frac{\mathbf{vv}}{c^2}\right)\cdot\mathbf{E} + \sum S\frac{q^2\mathbf{v}}{\gamma m}\times\mathbf{B}}.$$

Contrast *classical current evolution* for species s:

$$\partial_t \mathbf{J}_\mathrm{s} + \nabla\cdot\mathcal{P}_\mathrm{s} = \frac{q_\mathrm{s}}{m_\mathrm{s}}(\sigma_\mathrm{s}\mathbf{E} + \mathbf{J}_\mathrm{s}\times\mathbf{B}).$$

Unlike for the classical case, relativistic current evolution is not closed.

The response to the electric field is nonlinear and unique for each particle.

In the relativistic and dusty case, the response of current must be summed over all particles.

Linear(ized) response of (relativistic) dust:

$$\begin{aligned}
\bar{\mathbf{J}} &= \widehat{\mathbf{J}} + \mathbb{A}\cdot\bar{\mathbf{E}},\\
\mathbb{A} &:= \sum_\mathrm{p} q_\mathrm{p}\beta_\mathrm{p}\bar{\gamma}_\mathrm{p}^{-1}\Pi_\mathrm{p}^{\vartheta} S(\mathbf{x}-\mathbf{x}_\mathrm{p}^0),\\
\widehat{\mathbf{J}} &:= \sum_\mathrm{p}(q_\mathrm{p}\widehat{\mathbf{v}}S) - \frac{\Delta t}{2}\nabla\cdot\sum_\mathrm{p} q_\mathrm{p}\widehat{\mathbf{v}}_\mathrm{p}\widehat{\mathbf{v}}_\mathrm{p}S,\\
\widehat{\mathbf{v}}_\mathrm{p}^0 &:= \bar{\gamma}_\mathrm{p}^{-1}\Pi_\mathrm{p}^{\vartheta}\cdot\mathbf{u}_\mathrm{p}^0,\\
\Pi_\mathrm{p}^{\vartheta} &:= \frac{\mathbb{I} - \boldsymbol{\Omega}_\mathrm{p}\times\mathbb{I} + \boldsymbol{\Omega}_\mathrm{p}\boldsymbol{\Omega}_\mathrm{p}}{1+|\boldsymbol{\Omega}_\mathrm{p}|^2},\\
\boldsymbol{\Omega}_\mathrm{p} &:= \beta_\mathrm{p}\mathbf{B}',\\
\beta_\mathrm{p} &:= \frac{q_\mathrm{p}\Delta t}{2m_\mathrm{p}}.
\end{aligned}$$

Take-home points:

- IMM is not particular to particle codes. IMM, Implicit Source, and the Fully Implicit scheme can be used with a Vlasov or (higher-moment) fluid model and are defined by which terms are implicit.
- IMM assumes that current responds classically to the electric field (but a relativistic pusher can still be appropriate to handle high-energy particles).
- For truly relativistic problems, use the Implicit Source method.

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