A proposal for a shock-capturing, h-adaptive numerical algorithm for 2-fluid collisionless plasmas.

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presented April 11, 2007
updated August 25, 2007

Outline

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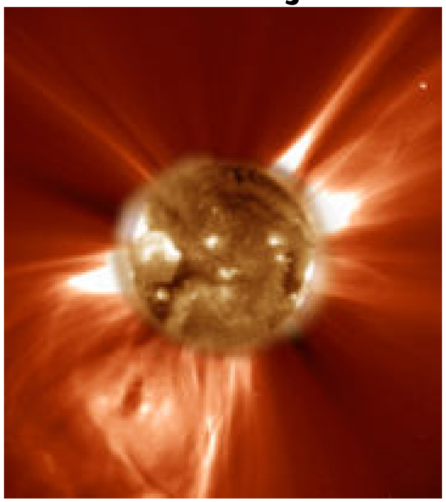
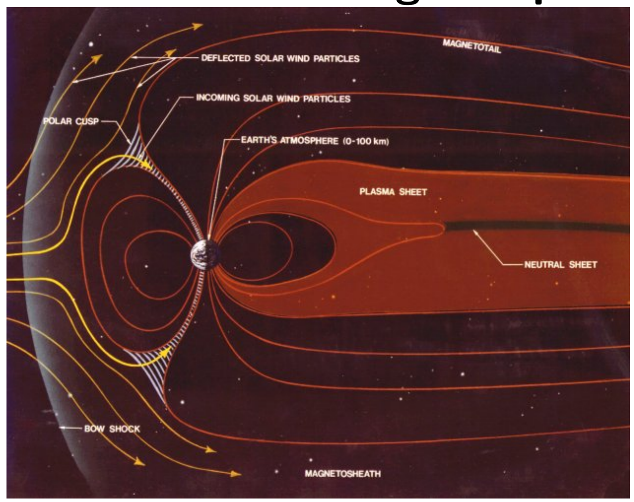
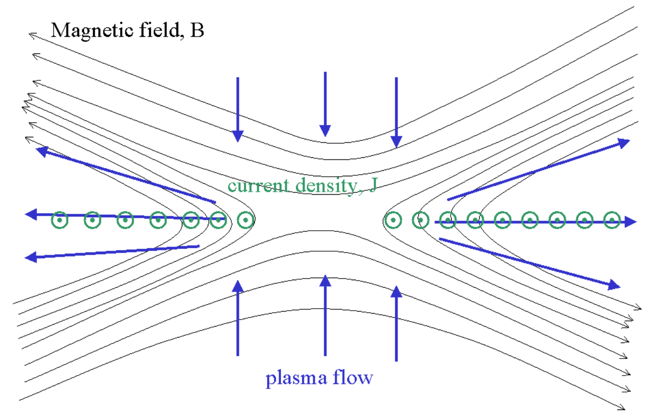
  (c) Goal: fast GEM

Overview: Space weather and magnetic reconnection.

Coronal Mass Ejection. **Earth's magnetosphere.** **Magnetic reconnection.**







① Magnetic reconnection leads to explosions of energy from the sun.

② Solar wind interacts with Earth's magnetosphere.

③ Magnetic reconnection releases energy in bow shock and magnetotail.

$\Longrightarrow$ Magnetic reconnection is critical to modeling space weather events.

Overview: What is a Plasma?

- A plasma is an ionized gas.
- As matter is heated, bonds are broken.
- **Phases of matter:**
  1. solid
  2. liquid (fixed bonds dissociated)
  3. gas (molecules dissociated)
  4. plasma (charges dissociated)
- Plasmas have free charges.
- So plasmas can conduct electrical current.
- So plasmas interact with electromagnetic fields.

Overview: Plasma model hierarchy

- **Three models of plasma:**
  ① **Kinetic model.** (Most accurate and highly expensive).
  Each species s is modeled by an evolving particle density function of phase space variables: $f_s(\mathbf{x}, \mathbf{v}, t)$
  ② **Two-fluid model.** Electrons and ions are modeled as distinct fluids flowing through one another. (Intermediate accuracy and expense).
  ③ **One-fluid model**, i.e. **Magnetohydrodynamics (MHD)**.
  (Least accurate & least expensive).
  - The ion motion constitutes the motion of the fluid.
  - The electron motion relative to the ions constitutes the current.
  - MHD assumes quasineutral and magnetostatic approximations.
- **Choice of model.**
  - Philosophy: use the least expensive model that exhibits the behavior of interest.
  - Need 2-fluid model to model fast reconnection. (Resistive MHD gives correct steady state, but is too slow by orders of magnitude.)

$\Longrightarrow$ Choose 2-fluid model.

Overview: Selective resolution

What makes our proposed 2-fluid solver unique?

- 2-fluid model admits fast waves (light waves, whistler waves).
  - Fast waves are needed for fast reconnection.
  - Fast waves are numerically expensive (require short time steps to satisfy CFL).
- Magnetic reconnection is usually a localized phenomenon.
- **Challenge: Can we selectively resolve fast waves only in regions where magnetic reconnection is occuring, and elsewhere use a coarser time step?**

  e.g. Can we use a 2-fluid model in reconnection regions and something more like MHD in the large majority of the domain?

Model equations: balance law framework.

The equations that govern a plasma model consist of the laws of electromagnetism plus conservation laws (conservation of mass, momentum, and energy).

We will generally write these evolution equations in the form of balance laws:

$$q_t + \nabla \cdot f(q) = s$$

where $q =$ state, $f =$ flux, and $s =$ source term.

Model equations: Laws of electromagnetism.

The fundamental laws of electromagnetism are:

① **Maxwell's equations:**

$$\underbrace{\partial_t \begin{bmatrix} c\mathbf{B} \\ \mathbf{E} \end{bmatrix} + c\nabla \times \begin{bmatrix} \mathbf{E} \\ -c\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{\epsilon_0}\mathbf{J} \end{bmatrix}}_{\text{evolution equations}} \text{ and } \underbrace{\nabla \cdot \begin{bmatrix} c\mathbf{B} \\ \mathbf{E} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\epsilon_0}\sigma \end{bmatrix},}_{\text{constraint equations}}$$

where $\mathbf{B}$ = magnetic field, $\mathbf{E}$ = electric field, σ = charge density, and $\mathbf{J}$ = current; and

② **The Lorentz force law:**

(a) particle version:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where q = charge, $\mathbf{v}$ = velocity of charge, and $\mathbf{F}$ = force on charge; or

(b) continuum version:

$$\mathbf{F} = \sigma(\mathbf{E} + \mathbf{J} \times \mathbf{B})$$

where σ = charge per volume, $\mathbf{J}$ = current per volume, and $\mathbf{F}$ = force per volume.

Model equations: Kinetic model

- The kinetic model is governed by the *Boltzmann equation*, which asserts that particle density is conserved as it flows through phase space:

$$\partial_t f_s + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f_s) + \nabla_{\mathbf{v}} \cdot (\mathbf{a_s} f_s) = \sum_p C_s^p[f_s, f_p]$$

- Independent quantities.
  - $\mathbf{x} =$ position
  - $\mathbf{v} = \dot{\mathbf{x}} =$ velocity
  - $t =$ time
- Parameters.
  - $s =$ species index (i=ion,e=electron)
  - $q_s =$ particle charge
  - $m_s =$ particle mass

- Dependent quantities and operators.
  - $f_s(\mathbf{x}, \mathbf{v}, t) =$ particle density function
  - $\mathbf{a}_s(\mathbf{x}, \mathbf{v}, t) = \dot{\mathbf{v}} = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) =$ acceleration
  - $C_s^p[f_s, f_p] =$ rate of change of f_s as a result of collisions with particles of species p. The collisionless plasma model assumes that this is zero.

- Taking moments of the Boltzmann equation yields balance laws for density, momentum, and energy of each species (i.e. the multifluid model).

Model equations: Two-fluid model

The general two-fluid model consists simply of gas dynamics for each of the two fluids, coupled to one another by drag force and heat transfer and coupled to Maxwell's equations by means of source terms consisting of the Lorentz force, the charge density, and the current and displacement currents. The gas dynamics equations are:

$$\boxed{\underbrace{\partial_t \begin{bmatrix} \rho_s \\ \rho_s \mathbf{v}_s \\ \mathcal{E}_s \end{bmatrix}}_{\text{conserved}} + \nabla \cdot \underbrace{\begin{bmatrix} \rho_s \mathbf{v}_s \\ \rho_s \mathbf{v}_s \mathbf{v}_s + p_s \mathbb{I} \\ \mathbf{v}_s (\mathcal{E}_s + p_s) \end{bmatrix}}_{\text{hyperbolic flux}} = \underbrace{\begin{bmatrix} 0 \\ \frac{q_s}{m_s} \rho_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) \\ \frac{q_s}{m_s} \rho_s \mathbf{v}_s \cdot \mathbf{E} \end{bmatrix}}_{\text{electromagnetic source}} + \nabla \cdot \underbrace{\begin{bmatrix} 0 \\ \underline{\underline{\sigma}}_s \\ \underline{\underline{\sigma}}_s \cdot \mathbf{v}_s + \kappa_s \nabla T_s \end{bmatrix}}_{\text{parabolic flux}} + \underbrace{\begin{bmatrix} 0 \\ \mathbf{R}_s \\ \mathbf{R}_s \cdot \mathbf{v}_s + Q_s \end{bmatrix}}_{\text{interactive source}}}$$

where $s = i$ (ion) or e (electron), $\frac{q_s}{m_s}$ is charge-to-mass ratio, ρ is mass density, $\mathbf{v}$ is fluid velocity, $\mathcal{E}$ is energy, p is pressure, $\underline{\underline{\sigma}}$ is viscous stress, and T is temperature.

A collisionless ideal plasma assumes that the interactive and parabolic fluxes are zero. This means that there is no direct coupling of the ions and the electrons; they only interact by means of the electromagnetic field. We also assume the ideal gas constitutive relations $\mathcal{E}_s = \frac{p_s}{\gamma_s - 1} + \frac{1}{2}\rho_s v_s^2$. The charge density and the current density of each species are given by the relations:

$$\sigma_s = \frac{q_s}{m_s}\rho_s \text{ and } \mathbf{J}_s = \frac{q_s}{m_s}\rho_s \mathbf{v}_s$$

Model equations: One-fluid model

The full system for the MHD equations is:

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\mathbf{v}\\ \tilde{\mathcal{E}}\\ \mathbf{B}\end{bmatrix} + \nabla\cdot\underbrace{\begin{bmatrix}\rho\mathbf{v}\\ \rho\mathbf{v}\mathbf{v}+\tilde{p}\,\underline{\underline{\delta}}-\frac{1}{\mu_0}\mathbf{B}\mathbf{B}\\ \mathbf{v}(\tilde{\mathcal{E}}+\tilde{p})-\frac{1}{\mu_0}\mathbf{B}\mathbf{B}\cdot\mathbf{v}\\ \mathbf{v}\mathbf{B}-\mathbf{B}\mathbf{v}\end{bmatrix}}_{\text{hyperbolic flux}} = \nabla\cdot\underbrace{\begin{bmatrix}0\\ \underline{\underline{\sigma}}\\ \underline{\underline{\sigma}}\cdot\mathbf{v}+\left(\kappa\nabla T+\eta\frac{1}{\mu_0^2}\nabla\cdot\left(\frac{1}{2}B^2\underline{\underline{\delta}}-\mathbf{B}\mathbf{B}\right)\right)\\ \eta\frac{1}{\mu_0}(\nabla\mathbf{B}^T-\nabla\mathbf{B})\end{bmatrix}}_{\text{parabolic flux}}$$

and $\nabla\cdot\mathbf{B}=0\,,$

where ρ is the mass density, $\mathbf{v}$ is the fluid velocity field, $\tilde{\mathcal{E}} := \mathcal{E} + \frac{1}{2\mu_0}B^2$ is the total energy (gas-dynamic energy plus magnetic energy), $\mathbf{B}$ is the magnetic field, and $\tilde{p} := p + \frac{1}{2\mu_0}B^2$ is the total pressure (gas-dynamic pressure plus magnetic pressure). The gas-dynamic pressure is $p = (\gamma-1)(\mathcal{E}-\frac{1}{2}\rho v^2)$, where γ is the ratio of specific heats.

For an ideal plasma we assume that the parabolic flux is zero, i.e. the viscous stress $\underline{\underline{\sigma}}$, the thermal conductivity κ, and the resistivity η are all taken to be zero.

Model equations: Nondimensionalization

Chosen characteristic values.

x_0 = typical length scale

v_0 = typical thermal velocity of an ion

n_0 = typical number density

B_0 = typical magnetic field strength

m_0 = mass of an ion

q_0 = charge strength of ion/electron

Immediate nondimensionalizations.

$$\mathbf{v}_s = v_0 \widehat{\mathbf{v}}_s$$

$$n_s = n_0 \widehat{n}_s$$

$$\mathbf{B} = B_0 \widehat{\mathbf{B}}$$

Implied nondimensionalizations.

$t = t_0 \widehat{t}$ where $t_0 := \frac{x_0}{v_0}$

$\partial_t = \frac{1}{t_0}\partial_{\widehat{t}}$

$\nabla = \frac{1}{x_0}\widehat{\nabla}$ where $\widehat{\nabla} := \nabla_{\widehat{\mathbf{x}}}$

$m_s = m_0 \widehat{m}_s$ where $m_0 := m_i$ and $\widehat{m}_s = \begin{cases} 1 & \text{if } s = i \\ \frac{m_e}{m_i} & \text{if } s = e \end{cases}$

$q_s = q_0 \widehat{q}_s$ where $q_0 := e$ and $\widehat{q}_s = \begin{cases} 1 & \text{if } s = i \\ -1 & \text{if } s = e \end{cases}$

$\rho_s = \rho_0 \widehat{\rho}_s$ where $\rho_0 := m_0 n_0$

$\sigma_s = \sigma_0 \widehat{\sigma}_s$ where $\sigma_0 := q_0 n_0$

$\mathbf{J}_s = J_0 \widehat{\mathbf{J}}_s$ where $J_0 := q_0 n_0 v_0$

$\mathbf{M}_s = M_0 \widehat{\mathbf{M}}_s$ where $M_0 := \rho_0 v_0 = m_0 n_0 v_0$

$p_s = p_0 \widehat{p}_s$ where $p_0 := \rho_0 v_0^2 = m_0 n_0 v_0^2$

$\mathcal{E}_s = \mathcal{E}_0 \widehat{\mathcal{E}}_s$ where $\mathcal{E}_0 := p_0$

$\mathbf{E} = E_0 \widehat{\mathbf{E}}$ where $E_0 := B_0 v_0$

Relationships:

$$\boxed{\begin{aligned} \widehat{\mathbf{J}}_i &= \widehat{\mathbf{M}}_i, & \widehat{\sigma}_i &= \widehat{n}_i = \widehat{\rho}_i, \\ -\widehat{\mathbf{J}}_e &= \frac{m_i}{m_e}\widehat{\mathbf{M}}_e, & -\widehat{\sigma}_e &= \widehat{n}_e = \frac{m_i}{m_e}\widehat{\rho}_e. \end{aligned}}$$

Model equations: Nondimensionalization

Nondimensionalized gas-dynamics.

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\rho}_s \\ \widehat{\mathbf{M}}_s \\ \widehat{\mathcal{E}}_s \end{bmatrix} + \widehat{\nabla}\cdot \begin{bmatrix} \widehat{\mathbf{M}}_s \\ \frac{\widehat{\mathbf{M}}_s \widehat{\mathbf{M}}_s}{\widehat{\rho}_s} + \widehat{p}_s \,\underline{\underline{\delta}} \\ \frac{\widehat{\mathbf{M}}_s}{\widehat{\rho}_s}\big(\widehat{\mathcal{E}}_s + \widehat{p}_s\big) \end{bmatrix} = \frac{1}{\widehat{r_L}} \begin{bmatrix} 0 \\ \frac{\widehat{q}_s}{\widehat{m}_s}(\widehat{\rho}_s \widehat{\mathbf{E}} + \widehat{\mathbf{M}}_s \times \widehat{\mathbf{B}}) \\ \frac{\widehat{q}_s}{\widehat{m}_s} \widehat{\mathbf{M}}_s \cdot \widehat{\mathbf{E}} \end{bmatrix}$$

where $\quad \widehat{p} = (\gamma - 1)\Big(\widehat{\mathcal{E}} - \dfrac{\widehat{M}^2}{2\widehat{\rho}}\Big)$

Here $r_L := \frac{m_0 v_0}{q_0 B_0}$ is the **Larmor radius**, the radius of curvature of the circular oscillation of a charge with characteristic values of mass and charge moving at the characteristic velocity perpendicular to the characteristic magnetic field. We define $\widehat{r_L} := \frac{r_L}{x_0} = \frac{m_0 v_0}{q_0 B_0 x_0}$.

Model equations: Nondimensionalized full system.

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\rho}_i \\ \widehat{\rho}_e \\ \widehat{\mathbf{M}}_i \\ \widehat{\mathbf{M}}_e \\ \widehat{\mathcal{E}}_i \\ \widehat{\mathcal{E}}_e \\ \widehat{\tilde{\mathbf{B}}} \\ \widehat{\mathbf{E}} \end{bmatrix} + \widehat{\nabla\cdot} \begin{bmatrix} \widehat{\mathbf{M}}_i \\ \widehat{\mathbf{M}}_e \\ \widehat{\mathbf{M}}_i\widehat{\mathbf{M}}_i/\widehat{\rho}_i + \widehat{p}_i\,\underline{\underline{\delta}} \\ \widehat{\mathbf{M}}_e\widehat{\mathbf{M}}_e/\widehat{\rho}_e + \widehat{p}_e\,\underline{\underline{\delta}} \\ (\widehat{\mathbf{M}}_i/\widehat{\rho}_i)\big(\widehat{\mathcal{E}}_i + \widehat{p}_i\big) \\ (\widehat{\mathbf{M}}_e/\widehat{\rho}_e)\big(\widehat{\mathcal{E}}_e + \widehat{p}_e\big) \\ -\widehat{c}\underline{\underline{\underline{\epsilon}}} \cdot \widehat{\mathbf{E}} \\ \widehat{c}\underline{\underline{\underline{\epsilon}}} \cdot \widehat{\tilde{\mathbf{B}}} \end{bmatrix} = \frac{1}{\widehat{r_L}} \begin{bmatrix} 0 \\ 0 \\ \widehat{\rho}_i\widehat{\mathbf{E}} + \widehat{\mathbf{M}}_i \times \widehat{\mathbf{B}} \\ -\frac{m_i}{m_e}(\widehat{\rho}_e\widehat{\mathbf{E}} + \widehat{\mathbf{M}}_e \times \widehat{\mathbf{B}}) \\ \widehat{\mathbf{M}}_i \cdot \widehat{\mathbf{E}} \\ -\frac{m_i}{m_e}\widehat{\mathbf{M}}_e \cdot \widehat{\mathbf{E}} \\ 0 \\ -\mathbf{J}/\widehat{\lambda_D}^2 \end{bmatrix}$$

where

$$\widehat{p}_i = (\gamma_i - 1)\Big(\widehat{\mathcal{E}}_i - \frac{\widehat{M}_i^2}{2\widehat{\rho}_i}\Big),$$

$$\widehat{p}_e = (\gamma_e - 1)\Big(\widehat{\mathcal{E}}_e - \frac{\widehat{M}_e^2}{2\widehat{\rho}_e}\Big),$$

$$\widehat{\mathbf{J}} = \widehat{\mathbf{J}}_i + \widehat{\mathbf{J}}_e = \widehat{\mathbf{M}}_i - \frac{m_i}{m_e}\widehat{\mathbf{M}}_e,$$

$$\widehat{\sigma} = \widehat{\sigma}_i + \widehat{\sigma}_e = \widehat{\rho}_i - \frac{m_i}{m_e}\widehat{\rho}_e,$$

$\widehat{\tilde{\mathbf{B}}} := \widehat{c}\widehat{\mathbf{B}}$,

$\underline{\underline{\underline{\epsilon}}}$ is the permutation tensor,

$\lambda_D := \sqrt{\frac{\epsilon_0 m_0 v_0^2}{n_0 q_0^2}}$, and

$$\widehat{\lambda_D}^2 := \frac{\lambda_D^2}{r_L^2} = \frac{\epsilon_0 B_0^2}{n_0 m_0}.$$

Here λ_D is the **Debye length**, which is the distance scale over which electrons screen out electric fields in plasmas (i.e. the distance scale over which significant charge separation can occur).

Numerical method: operator splitting.

We aim for second-order accuracy. This justifies operator splitting.

1. ODE solver
   (a) Backward Differencing for interdependent components.
   (b) Energy solver.
2. Hyperbolic PDE solver
   (a) Gas-dynamics solver (explicit, shock-capturing)
   (b) Maxwell solver (ultimately implicit)

Numerical method: ODE solver.

We need to solve the ODE:

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\rho}_i \\ \widehat{\rho}_e \\ \widehat{\mathbf{M}}_i \\ \widehat{\mathbf{M}}_e \\ \widehat{\mathcal{E}}_i \\ \widehat{\mathcal{E}}_e \\ \widehat{\mathbf{B}} \\ \widehat{\mathbf{E}} \end{bmatrix} = \frac{1}{\widehat{r_L}} \begin{bmatrix} 0 \\ 0 \\ \widehat{\rho}_i \widehat{\mathbf{E}} + \widehat{\mathbf{M}}_i \times \widehat{\mathbf{B}} \\ -\frac{m_i}{m_e}(\widehat{\rho}_e \widehat{\mathbf{E}} + \widehat{\mathbf{M}}_e \times \widehat{\mathbf{B}}) \\ \widehat{\mathbf{M}}_i \cdot \widehat{\mathbf{E}} \\ -\frac{m_i}{m_e} \widehat{\mathbf{M}}_e \cdot \widehat{\mathbf{E}} \\ 0 \\ \frac{-1}{\widehat{\lambda_D}^2}\big(\widehat{\mathbf{M}}_i - \frac{m_i}{m_e}\widehat{\mathbf{M}}_e\big) \end{bmatrix}$$

Strategy: decouple interdependent components from others.

Numerical method: ODE solver.

Interdependent components of ODE.

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\mathbf{M}}_i \\ \widehat{\mathbf{M}}_e \\ \widehat{\mathbf{E}} \end{bmatrix} = \frac{1}{\widehat{r_L}} \begin{bmatrix} \widehat{\rho}_i \widehat{\mathbf{E}} + \widehat{\mathbf{M}}_i \times \widehat{\mathbf{B}} \\ -\frac{m_i}{m_e}(\widehat{\rho}_e \widehat{\mathbf{E}} + \widehat{\mathbf{M}}_e \times \widehat{\mathbf{B}}) \\ \left(\frac{m_i}{m_e}\widehat{\mathbf{M}}_e - \widehat{\mathbf{M}}_i\right) / \widehat{\lambda_D}^2 \end{bmatrix}$$

This is an ODE with constant coefficients and imaginary eigenvalues!

Use the **TR-BDF2** method. This is an implicit, 2-stage Runge-Kutta method based on taking a half time-step with the Trapezoidal Rule (TR) and then a half step with the 2-step BDF (Backward Differentiation Formula) method:

$$U^* = U^n + \frac{k}{4}(f(U^n) + f(U^*)),$$

$$3U^{n+1} - 4U^* + U^n = kf(U^{n+1}).$$

Numerical method: ODE solver.

Dependent components of ODE.

The evolution of the energies is determined from the interdependent components:

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\mathcal{E}}_i \\ \widehat{\mathcal{E}}_e \end{bmatrix} = \frac{1}{\widehat{r_L}} \begin{bmatrix} \widehat{\mathbf{M}}_i \cdot \widehat{\mathbf{E}} \\ -\frac{m_i}{m_e}\widehat{\mathbf{M}}_e \cdot \widehat{\mathbf{E}} \end{bmatrix}$$

A second-order solver for each of these energy variables is:

$$\frac{\widehat{\mathcal{E}}_s^{n+1} - \widehat{\mathcal{E}}_s^{n}}{\Delta t} = \frac{(\widehat{\mathbf{J}}_s \cdot \widehat{\mathbf{E}})^n + (\widehat{\mathbf{J}}_s \cdot \widehat{\mathbf{E}})^{n+1}}{2\widehat{r_L}}$$

Numerical method: Hyperbolic PDE solver.

The hyperbolic part decouples into three independent systems:

1. Gas-dynamics for ions
2. Gas-dynamics for electrons
3. Homogeneous Maxwell's equations

Numerical method: Hyperbolic PDE solver.

Gas dynamics solver.

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\rho}_i \\ \widehat{\rho}_e \\ \widehat{\mathbf{M}}_i \\ \widehat{\mathbf{M}}_e \\ \widehat{\mathcal{E}}_i \\ \widehat{\mathcal{E}}_e \end{bmatrix} + \widehat{\nabla\cdot} \begin{bmatrix} \widehat{\mathbf{M}}_i \\ \widehat{\mathbf{M}}_e \\ (\widehat{\mathbf{M}}_i \widehat{\mathbf{M}}_i)/\widehat{\rho}_i + \widehat{p}_i \, \underline{\underline{\delta}} \\ (\widehat{\mathbf{M}}_e \widehat{\mathbf{M}}_e)/\widehat{\rho}_e + \widehat{p}_e \, \underline{\underline{\delta}} \\ (\widehat{\mathbf{M}}_i/\widehat{\rho}_i)\big(\widehat{\mathcal{E}}_i + \widehat{p}_i\big) \\ (\widehat{\mathbf{M}}_e/\widehat{\rho}_e)\big(\widehat{\mathcal{E}}_e + \widehat{p}_e\big) \end{bmatrix} = 0$$

For the gas-dynamics solver we plan to use a standard explicit finite-volume shock-capturing method: Godunov/Roe solver with higher-order corrections.

Numerical method: Hyperbolic PDE solver.

Homogeneous Maxwell solver. The Maxwell solver "merely" solves the constant-coefficient hyperbolic homogeneous system:

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\tilde{\mathbf{B}}} \\ \widehat{\mathbf{E}} \end{bmatrix} + \widehat{c}\widehat{\nabla\times} \begin{bmatrix} \widehat{\mathbf{E}} \\ -\widehat{\tilde{\mathbf{B}}} \end{bmatrix} = 0,$$

where $\widehat{\tilde{\mathbf{B}}} := \widehat{c}\widehat{\mathbf{B}}$ and $\epsilon := \widehat{\lambda_D}^2 \widehat{r_L}$

[FIX: write down eigenvalues and eigenvectors.]

This equation supports waves propagating at the speed of light. An explicit solver requires a short time step[1], so use an **implicit method**.

Two formidable expected challenges in regions where we do not resolve fast waves:

① Will the matrix in our implicit method be ill-conditioned?

② Will we compute physical solutions?

[1] or a large stencil, which is prohibitively expensive in multiple dimensions

Model equations: Divergence constraints.

Problem: To ensure that the solution remains physical, we need to enforce the divergence constraints:

$$\widehat{\nabla} \cdot \begin{bmatrix} \widehat{\tilde{\mathbf{B}}} \\ \widehat{\mathbf{E}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\epsilon}\widehat{\sigma} \end{bmatrix}$$

One solution: Switch to a **potential formulation**.

Recall Maxwell's full system of equations:

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\tilde{\mathbf{B}}} \\ \widehat{\mathbf{E}} \end{bmatrix} + \widehat{c}\,\widehat{\nabla\times} \begin{bmatrix} \widehat{\mathbf{E}} \\ -\widehat{\tilde{\mathbf{B}}} \end{bmatrix} = \begin{bmatrix} 0 \\ -\widehat{\mathbf{J}}/\epsilon \end{bmatrix},$$

$$\widehat{\nabla} \cdot \begin{bmatrix} \widehat{\tilde{\mathbf{B}}} \\ \widehat{\mathbf{E}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\epsilon}\widehat{\sigma} \end{bmatrix}, \qquad \begin{aligned} \widehat{\tilde{\mathbf{B}}} &:= \widehat{c}\widehat{\mathbf{B}}, \text{ and} \\ \epsilon &:= \widehat{\lambda_D}^2 \widehat{r_L}. \end{aligned}$$

Drop hats. We will rewrite this system in terms of vector potentials.

Model equations: Potential formulation.

Potential formulation of Maxwell's equations.

Use the homogeneous equations

$$\nabla \cdot \mathbf{B} = 0,$$
$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$$

to write

$$\mathbf{B} = \nabla \times \mathbf{A},$$
$$\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}.$$

Substituting these representations into the nonhomogeneous equations,

$$\sigma/\epsilon = \nabla \cdot \mathbf{E} \quad \text{and}$$
$$\mathbf{J}/\epsilon = -\partial_t \mathbf{E} + c^2 \nabla \times \mathbf{B},$$

and imposing the generic gauge condition $\nabla \cdot \mathbf{A} = D$, gives:

$$-\frac{\sigma}{\epsilon} = \nabla^2\phi + \partial_t D,$$
$$\frac{\mathbf{J}}{\epsilon} = \partial_{tt}\mathbf{A} - c^2\nabla^2\mathbf{A} + \nabla(\partial_t\phi + c^2 D).$$

The drift from the gauge condition satisfies the wave equation, so it should disperse:

$$0 = \partial_{tt}(\nabla \cdot \mathbf{A} - D) - c^2\nabla^2(\nabla \cdot \mathbf{A} - D).$$

Select the Lorentz gauge condition, $D = -\partial_t\phi/c^2$. Maxwell's equations become wave equations:

$$\frac{\sigma}{\epsilon} = \frac{1}{c^2}\partial_{tt}\phi - \nabla^2\phi$$
$$\frac{\mathbf{J}}{\epsilon} = \partial_{tt}\mathbf{A} - c^2\nabla^2\mathbf{A}$$

Model equations: Potential formulation.

Four-vector potential.

To write this potential formulation as a single equation, define the **4-vector potential** A^μ and the **current density 4-vector** J^μ by:

$$A^\mu := \begin{bmatrix} \phi/c \\ \mathbf{A} \end{bmatrix}, \quad J^\mu := \begin{bmatrix} c\sigma \\ \mathbf{J} \end{bmatrix}.$$

Then

$$\partial_{tt} A^\mu - c^2 \nabla^2 A^\mu = J^\mu / \epsilon.$$

The wave equation for the potential implies a first-order system for the time and space derivatives of the potential:

$$\partial_t \begin{bmatrix} \partial_t A^\mu \\ c\nabla A^\mu \end{bmatrix} - c\nabla \cdot \begin{bmatrix} c\nabla A^\mu \\ \partial_t A^\mu \underline{\underline{\delta}} \end{bmatrix} = \begin{bmatrix} J^\mu/\epsilon \\ 0 \end{bmatrix}$$

These derivatives of the potential specify the electromagnetic field by the relations:

$$\mathbf{B} = \nabla \times \mathbf{A} = \underline{\underline{\underline{\epsilon}}} : \nabla \mathbf{A},$$

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla \phi = -\partial_t \mathbf{A} - \nabla A^0$$

For initial conditions, imposing

$$0 = (\partial_t \phi)|_{t=0} = (\nabla \phi)|_{t=0},$$

$$\text{i.e.,} \quad \begin{bmatrix} \partial_t A^0 \\ c\nabla A^0 \end{bmatrix}_{t=0} = 0,$$

gives

$$\begin{bmatrix} \partial_t \mathbf{A} \\ c\nabla \mathbf{A} \end{bmatrix}_{t=0} = \begin{bmatrix} -\mathbf{E} \\ -c\nabla\nabla^{-2}\nabla \times \mathbf{B} \end{bmatrix}.$$

Model equations: Potential formulation, full system.

The full system in the potential formulation consists of 10 gas-dynamics equations and 16 equations for the electromagnetic potential:

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\rho}_i \\ \widehat{\rho}_e \\ \widehat{\mathbf{M}}_i \\ \widehat{\mathbf{M}}_e \\ \widehat{\mathcal{E}}_i \\ \widehat{\mathcal{E}}_e \\ \widehat{V^\mu} \\ \widehat{\mathbf{W}^\mu} \end{bmatrix} + \widehat{\nabla}\cdot \begin{bmatrix} \widehat{\mathbf{M}}_i \\ \widehat{\mathbf{M}}_e \\ \widehat{\mathbf{M}}_i\widehat{\mathbf{M}}_i/\widehat{\rho}_i + \widehat{p}_i\,\underline{\underline{\delta}} \\ \widehat{\mathbf{M}}_e\widehat{\mathbf{M}}_e/\widehat{\rho}_e + \widehat{p}_e\,\underline{\underline{\delta}} \\ (\widehat{\mathbf{M}}_i/\widehat{\rho}_i)\big(\widehat{\mathcal{E}}_i + \widehat{p}_i\big) \\ (\widehat{\mathbf{M}}_e/\widehat{\rho}_e)\big(\widehat{\mathcal{E}}_e + \widehat{p}_e\big) \\ -\widehat{c}\widehat{\mathbf{W}^\mu} \\ -\widehat{c}\widehat{V^\mu}\underline{\underline{\delta}} \end{bmatrix} = \frac{1}{\widehat{r_L}} \begin{bmatrix} 0 \\ 0 \\ \widehat{\rho}_i\widehat{\mathbf{E}} + \widehat{\mathbf{M}}_i \times \widehat{\mathbf{B}} \\ -\frac{m_i}{m_e}(\widehat{\rho}_e\widehat{\mathbf{E}} + \widehat{\mathbf{M}}_e \times \widehat{\mathbf{B}}) \\ \widehat{\mathbf{M}}_i \cdot \widehat{\mathbf{E}} \\ -\frac{m_i}{m_e}\widehat{\mathbf{M}}_e \cdot \widehat{\mathbf{E}} \\ \widehat{J^\mu}/\widehat{\lambda_D}^2 \\ 0 \end{bmatrix}$$

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with constitutive relations:

$$\widehat{p}_i = (\gamma_i - 1)\Big(\widehat{\mathcal{E}}_i - \frac{\widehat{M_i^2}}{2\widehat{\rho}_i}\Big),$$

$$\widehat{p}_e = (\gamma_e - 1)\Big(\widehat{\mathcal{E}}_e - \frac{\widehat{M_e^2}}{2\widehat{\rho}_e}\Big),$$

with ancillary definitions:

$$\widehat{V^\mu} := \partial_t \widehat{A^\mu},$$

$$\widehat{\mathbf{W}^\mu} := c\widehat{\nabla}\widehat{A^\mu},$$

with defining relations:

$$\widehat{J^\mu} = \begin{bmatrix} \widehat{c}\,\widehat{\sigma} \\ \widehat{\mathbf{J}} \end{bmatrix},$$

$$\widehat{\sigma} = \widehat{\sigma}_i + \widehat{\sigma}_e = \widehat{\rho}_i - \frac{m_i}{m_e}\widehat{\rho}_e,$$

$$\widehat{\mathbf{J}} = \widehat{\mathbf{J}}_i + \widehat{\mathbf{J}}_e = \widehat{\mathbf{M}}_i - \frac{m_i}{m_e}\widehat{\mathbf{M}}_e,$$

$$\widehat{c}\widehat{\mathbf{B}}^i = \epsilon^{ij}{}_k \widehat{W}^k_j,$$

$$\widehat{\mathbf{E}} = -\widehat{\mathbf{W}}^0 - \widehat{\mathbf{V}},$$

and with initial conditions:

$$\begin{bmatrix} \widehat{V}^0 \\ \widehat{\mathbf{W}}^\mathbf{0} \end{bmatrix}_0 = 0 \quad \text{and}$$

$$\begin{bmatrix} \widehat{V}^i \\ \widehat{\mathbf{W}}^i \end{bmatrix}_0 = \begin{bmatrix} -\widehat{\mathbf{E}}^i \\ -\widehat{\nabla}\widehat{\nabla^{-2}}[\widehat{\nabla\times}(\widehat{c}\widehat{\mathbf{B}})]^i \end{bmatrix}_0$$

Numerical method: potential formulation.

For the potential formulation the numerical method is the same as the electromagnetic field formulation with modifications in the following:

① ODE solver: interdependent components, and

② hyperbolic solver: wave equation solver.

Numerical method: ODE solver (potential formulation).

Constant components of ODE.

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\rho}_i \\ \widehat{\rho}_e \\ \widehat{\mathbf{W}^\mu} \end{bmatrix} = 0$$

Interdependent components of ODE.

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\mathbf{M}}_i \\ \widehat{\mathbf{M}}_e \\ \widehat{\mathbf{V}} \end{bmatrix} = \frac{1}{\widehat{r_L}} \begin{bmatrix} \widehat{\rho}_i \widehat{\mathbf{E}} + \widehat{\mathbf{M}}_i \times \widehat{\mathbf{B}} \\ -\frac{m_i}{m_e}(\widehat{\rho}_e \widehat{\mathbf{E}} + \widehat{\mathbf{M}}_e \times \widehat{\mathbf{B}}) \\ (\widehat{\mathbf{M}}_i - \frac{m_i}{m_e}\widehat{\mathbf{M}}_e)/\widehat{\lambda_D}^2 \end{bmatrix},$$

where $\widehat{\mathbf{E}} = -\widehat{\mathbf{W}}^0 - \widehat{\mathbf{V}}$,

This is the same constant-coefficient ODE as in the electromagnetic formulation, except that it has an additional constant source term from $\widehat{\mathbf{W}}^0$.

ODE component dependent only on constants.

$$\partial_{\widehat{t}} \left[\widehat{V}^0\right] = \frac{1}{\widehat{r_L}} \left[\widehat{c}(\widehat{\rho}_i - \tfrac{m_i}{m_e}\widehat{\rho}_e)/\widehat{\lambda_D}^2\right]$$

ODE components dependent on interdependent components.

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{\mathcal{E}}_i \\ \widehat{\mathcal{E}}_e \end{bmatrix} = \frac{1}{\widehat{r_L}} \begin{bmatrix} \widehat{\mathbf{M}}_i \cdot \widehat{\mathbf{E}} \\ -\frac{m_i}{m_e}\widehat{\mathbf{M}}_e \cdot \widehat{\mathbf{E}} \end{bmatrix}$$

Numerical method: hyperbolic solver (potential formulation).

(The gas-dynamic solver remains unchanged.)

Homogeneous potential solver.

$$\partial_{\widehat{t}} \begin{bmatrix} \widehat{V^\mu} \\ \widehat{\mathbf{W}^\mu} \end{bmatrix} + \widehat{\nabla\cdot} \begin{bmatrix} -\widehat{c}\widehat{\mathbf{W}^\mu} \\ -\widehat{c}\widehat{V^\mu}\underline{\underline{\delta}} \end{bmatrix} = 0$$

[FIX: write down eigenvectors.]

Planned work.

Improvements on initial implementation:

- Not maintaining divergence condition $\longrightarrow$ potential formulation or divergence cleaning.
- Explicit $\longrightarrow$ implicit Maxwell's solver.
- 1D $\longrightarrow$ 2D model.

Goal: a fast solution to the Geospace Environmental Modeling (GEM) Magnetic Reconnection Challenge.

References

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