HW 3, due Sep 22

(1) $\vec{r}(t) = \begin{pmatrix} 4\sin t \\ 3t \\ 4\cos t \end{pmatrix}$, $0 \le t \le 3\pi$

Preliminary calculations

$\vec{r}'(t) = \begin{pmatrix} 4\cos t \\ 3 \\ -4\sin t \end{pmatrix}$

$|\vec{r}'(t)| = 5$

$\vec{r}''(t) = \begin{pmatrix} -4\sin t \\ 0 \\ -4\cos t \end{pmatrix}$

$\vec{r}' \times \vec{r}'' = \begin{pmatrix} -12\cos t \\ 16 \\ 12\sin t \end{pmatrix}$

$|\vec{r}' \times \vec{r}''| = 20$

$\vec{r}'''(t) = \begin{pmatrix} -4\cos t \\ 0 \\ 4\sin t \end{pmatrix}$

(2) (i) length of curve (L)

$L = \int_0^{3\pi} \frac{ds}{dt}\, dt$

$= \int_0^{3\pi} |\vec{r}'|\, dt$

$= \int_0^{3\pi} 5\, dt$

$\boxed{L = 15\pi}$ (2)

(4) (ii) curvature & torsion

$\kappa = \frac{|r' \times r''|}{|r'|^3}$

$= \frac{20}{5^3}$

$\boxed{\kappa = \frac{4}{25}}$ (2)

$\tau = \frac{(r' \times r'') \cdot r'''}{|r' \times r''|^2}$

$= \frac{48}{20^2} = \frac{\cancel{4^2} \cdot 3}{\cancel{4^2} \cdot 5^2}$

$\boxed{\tau = \frac{3}{25}}$ (2)

(5) TNB

(iii) unit tangent

$\hat{T} = \frac{r'}{|r'|}$

$\boxed{\hat{T} = \frac{1}{5}\begin{pmatrix} 4\cos t \\ 3 \\ -4\sin t \end{pmatrix}}$ (1)

binormal

$\hat{B} = \frac{r' \times r''}{|r' \times r''|}$

$\boxed{\hat{B} = \frac{1}{5}\begin{pmatrix} -3\cos t \\ 4 \\ 3\sin t \end{pmatrix}}$ (2)

principal unit normal

$\hat{N} = \hat{B} \times \hat{T}$

$= \frac{1}{25}\begin{pmatrix} 25\sin t \\ 0 \\ 25\cos t \end{pmatrix}$

$\boxed{\hat{N} = \begin{pmatrix} -\sin t \\ 0 \\ -\cos t \end{pmatrix}}$ (2)

(3) *(iv) eqn. for osculating plane at time t of the form

$Ax + By + Cz = D$.

Sol. A normal vector is

$\vec{n} := \begin{pmatrix} A \\ B \\ C \end{pmatrix} := 5\hat{B} = \begin{pmatrix} -3\cos t \\ 4 \\ 3\sin t \end{pmatrix}$

Equation of plane at time t is

$\vec{n}(t) \cdot \vec{r} = \vec{n}(t) \cdot \vec{r}(t)$,

i.e. $\begin{pmatrix} A \\ B \\ C \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \cdot \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$,

i.e. $\begin{pmatrix} -3\cos t \\ 4 \\ 3\sin t \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3\cos t \\ 4 \\ 3\sin t \end{pmatrix} \cdot \begin{pmatrix} 4\sin t \\ 3t \\ 4\cos t \end{pmatrix}$,

i.e. $\boxed{(-3\cos t)x + 4y + (3\sin t)z = 12t}$

(3)

(3) *(v) Show that the line through any point on the curve parallel to the principle unit normal intersects the y-axis with angle $\pi/2$.

Solution A parametrization for the point at time t_0 is

$\tilde{r}(t) = \vec{r}(t_0) + t\hat{N}(t_0)$

$= \begin{pmatrix} 4\sin t_0 \\ 3t \\ 4\cos t_0 \end{pmatrix} + t\begin{pmatrix} -\sin t_0 \\ 0 \\ -\cos t_0 \end{pmatrix}$

So $\tilde{r}(4) = \begin{pmatrix} 0 \\ 3t \\ 0 \end{pmatrix}$,

which is on the y axis, and

$\cos\theta = \hat{N} \cdot \hat{J} = 0 \Rightarrow \boxed{\theta = \frac{\pi}{2}}$

(2) (vi) angle of tangent line with y-axis is given by

$\cos\theta = \hat{T} \cdot \hat{J}$

$= \frac{3}{5}$ (a constant)

(2) (vii) Find arclength parametrization

$S(t) = \int_0^t |r'(t)|\, dt$

$= 5t$.

So $t = s/5$.

So $\vec{R}(s) = \vec{r}(s/5)$

$\vec{R}(s) = \begin{pmatrix} 4\sin s/5 \\ 3s/5 \\ 4\cos s/5 \end{pmatrix}$

(2) (viii) Center and radius of osculating circle:

$\rho = \frac{1}{\kappa} = \frac{25}{4}$ (radius)

$\text{center}(t) = \vec{r}(t) + \rho\hat{N}(t)$

$= \begin{pmatrix} 4\sin t \\ 3t \\ 4\cos t \end{pmatrix} + \frac{25}{4}\begin{pmatrix} -\sin t \\ 0 \\ -\cos t \end{pmatrix}$

(1) (ix) The answer to part (viii) is a parametrization of the locus of the centers of curvature. (This locus is called the evolute of the curve). (A locus is a set of points which satisfy a particular condition.)

$\boxed{\text{center}(t) = \begin{pmatrix} -9/4\sin t \\ 3t \\ -9/4\cos t \end{pmatrix}}$

(2)

$$\vec{r}(\theta) = \rho(\theta)\,\hat{r}(\theta),$$

where $\hat{r}(\theta) := \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

$$\boxed{r' = \rho'\hat{r} + \rho\hat{r}'},$$

where $\hat{r}' = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

Notice: $\hat{r}\cdot\hat{r}' = 0$,

$|\hat{r}'| = 1$

(a) $|r'|^2 = (\rho')^2 + \rho^2$

$L = \int_a^b |r'|\,dt$

$$\boxed{L = \int \sqrt{(\rho')^2 + \rho^2}\,dt}$$

(b) $\vec{r}'' = (\vec{r}')'$

$= (\rho'\hat{r} + \rho\hat{r}')'$

$= \rho''\hat{r} + 2\rho'\hat{r}' + \rho\hat{r}''$

[But $\hat{r}'' = -\hat{r}$]

so $\vec{r}'' = (\rho'' - \rho)\hat{r} + 2\rho'\hat{r}'$

So $\vec{r}' \times \vec{r}'' = (\rho'\hat{r} + \rho\hat{r}') \times ((\rho'' - \rho)\hat{r} + 2\rho'\hat{r}')$

$= 2\rho'^2(\hat{r}\times\hat{r}') + \rho(\rho'' - \rho)(\hat{r}'\times\hat{r})$

$= [2(\rho')^2 + \rho(\rho - \rho'')](\hat{r}\times\hat{r}')$.

So $|\vec{r}'\times\vec{r}''| = |2(\rho')^2 + \rho(\rho - \rho'')|$.

So $$\boxed{K = \frac{|\vec{r}'\times\vec{r}''|}{|r'|^3} = \frac{|2(\rho')^2 + \rho(\rho - \rho'')|}{(\rho^2 + (\rho')^2)^{3/2}}}$$

(3) $\vec{r}(s) = \begin{pmatrix} a + \int_0^s \cos(\nu(\alpha))\,d\alpha \\ b + \int_0^s \sin(\nu(\alpha))\,d\alpha \end{pmatrix}$,

where $\nu(s) = \int_0^s K(\sigma)\,d\sigma$

Show curvature$(s) = K(s)$.

<u>Proof</u>.

Recall curvature$(s) = \dfrac{|r'\times r''|}{|r'|^3}$.

$\vec{r}' = \begin{pmatrix} \cos\nu \\ \sin\nu \end{pmatrix}$, by 1^{st} FTC.

$\vec{r}'' = K\begin{pmatrix} -\sin\nu \\ \cos\nu \end{pmatrix}$ by 1^{st} FTC.

$\vec{r}'\times\vec{r}'' = \begin{pmatrix} \cos\nu \\ \sin\nu \\ 0 \end{pmatrix} \times K\begin{pmatrix} -\sin\nu \\ \cos\nu \\ 0 \end{pmatrix}$

$= K\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$|r'| = 1$. $|r'\times r''| = K$.

So curvature$(s) = K(s)$.

(4) Suppose $\vec{r}(t)$ has nonzero curvature but zero torsion. Show $\vec{r}$ lies in a plane.

<u>Proof</u>.

$$\frac{d\hat{B}}{ds} = -\tau\hat{N} = 0.$$

* So $\hat{B}(s) = \hat{B}(0) =: \hat{B}_0$.

So $(\vec{r}\cdot\hat{B}_0)' = \vec{r}'\cdot\hat{B}_0 = s'\hat{T}\cdot\hat{B} = 0$

So $\vec{r}(t)\cdot\hat{B}_0 = \text{constant} =: D$.

So $\vec{r}(t)$ lies in the plane

$\vec{r}\cdot\hat{B}_0 = D$. Done.

Question: where did we use that $\vec{r}(t)$ has nonzero curvature?