Homework 9, due Tue Nov 4
Section 14.8

(17) Find the point on the plane $\vec{n}\cdot\vec{r} = D$ closest to the point $\vec{r}_0 = (1,1,1)$, where $\vec{n} = (1,2,3)$ and $D = 13$.

Solution

Let $g(\vec{r}) = \vec{n}\cdot\vec{r}$ and let

$f(\vec{r}) = \|\vec{r}-\vec{r}_0\|^2 = (\vec{r}-\vec{r}_0)\cdot(\vec{r}-\vec{r}_0)$.

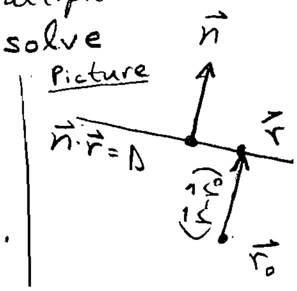
We seek to minimize f (equivalently $\sqrt{f}$) given the constraint $g(\vec{r}) = D$.

Use Lagrange multipliers.

So we seek to solve

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = D \end{cases}$$

Picture



$\nabla f = 2(\vec{r}-\vec{r}_0)$.

$\nabla g = \vec{n}$.

So we need to solve

$$\begin{cases} \vec{r}-\vec{r}_0 = \tilde{\lambda}\,\vec{n} \\ \vec{n}\cdot\vec{r} = D \end{cases} \qquad (\tilde{\lambda} := \lambda/2)$$

Take the dot product of the first equation with $\vec{n}$:

$$\underbrace{\vec{n}\cdot\vec{r}}_{D} - \vec{n}\cdot\vec{r}_0 = \tilde{\lambda}\,\vec{n}\cdot\vec{n}$$

So $\tilde{\lambda} = \dfrac{D - \vec{n}\cdot\vec{r}_0}{\vec{n}\cdot\vec{n}}$

So $$\vec{r} = \vec{r}_0 + \left(\frac{D-\vec{n}\cdot\vec{r}_0}{\vec{n}\cdot\vec{n}}\right)\vec{n}$$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix} + \left(\frac{13-6}{14}\right)\cdot\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\boxed{\vec{r} = \left(\tfrac{3}{2}, 2, \tfrac{5}{2}\right)}$$

(18) Find the point on the sphere $\|\vec{r}\|^2 = R^2$, where $R=2$, farthest from the point $\vec{r}_0 = (1,-1,1)$.

Solution

Let $g(\vec{r}) = \|\vec{r}\|^2 = r\cdot r$, and let

$f(\vec{r}) = \|r - r_0\|^2 = (\vec{r}-\vec{r}_0)\cdot(\vec{r}-\vec{r}_0)$.

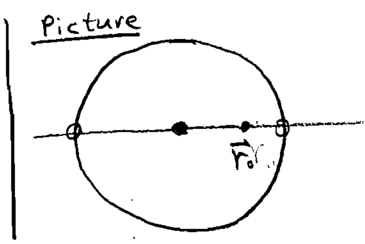
We seek to minimize f (equivalently $\sqrt{f}$) given the constraint $g(\vec{r}) = D$.

Use Lagrange multipliers.

So we seek to solve

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = R^2 \end{cases}$$

Picture



$\nabla f = 2(\vec{r}-\vec{r}_0)$

$\nabla g = 2\vec{r}$

So we need to solve

$$\begin{cases} \vec{r}-\vec{r}_0 = \lambda\vec{r} \\ \vec{r}\cdot\vec{r} = R^2 \end{cases}$$

The first equation says that $\vec{r}$ is in (the opposite of) the direction of $\vec{r}_0$:

$$\vec{r} = \underbrace{\left(\frac{1}{1-\lambda}\right)}_{\text{Call } \mu.}\vec{r}_0$$

So we need to solve

$$\begin{cases} \vec{r} = \mu\vec{r}_0 \\ \vec{r}\cdot\vec{r} = R^2 \end{cases}$$

So $\mu^2\|r_0\|^2 = R^2$

So $\mu = \dfrac{\pm R}{\|\vec{r}_0\|}$

So $$\boxed{\begin{aligned}\vec{r} &= \pm\frac{R}{\|\vec{r}_0\|}\vec{r}_0 \\ &= \pm\frac{2}{\sqrt{3}}(1,-1,1)\end{aligned}}$$

$f(\vec{r}) = \left\|\left(\pm\frac{R}{\|r_0\|} - 1\right) r_0\right\|$

max: $\|r_0\| + R = \sqrt{3} + 2$

min: $\left|\|r_0\| - R\right| = 2 - \sqrt{3}$

Farthest point: $\left(\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\right)$

(Section 14.8)

(26) Maximize $f = xyz$
given $g := x+y+z^2 = 16$, x,y,z positive.

Solution

Need $\begin{cases} \nabla f = \lambda \nabla g \\ g = 16. \end{cases}$

i.e. $\begin{cases} \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 2z \end{pmatrix} \\ x+y+z^2 = 16 \end{cases}$

Divide first two equations. $\begin{pmatrix} z \neq 0, \\ \lambda \neq 0 \end{pmatrix}$

Get $x = y$. Eliminates one equation.

So $\begin{cases} xz = \lambda \\ x^2 = \lambda 2z \\ 2x + z^2 = 16 \end{cases}$

Eliminate λ:

$\begin{cases} x^2 = 2xz^2 \ (\Rightarrow x = 2z^2) \\ 2x + z^2 = 16 \end{cases}$

So $5z^2 = 16$

$$\boxed{\begin{aligned} z &= \frac{4}{\sqrt{5}} \\ x = y &= \frac{32}{5} \end{aligned}}$$

$f(x,y,z) = xyz = \frac{32}{5} \cdot \frac{32}{5} \cdot \frac{4}{\sqrt{5}}$

$= \frac{4096}{25\sqrt{5}}$

(27) $V = 8xyz$, $g = x^2+y^2+z^2 = 1$,
maximize V.

Solution

Need $\begin{cases} \nabla V = \tilde{\lambda} \nabla g \\ g = 1 \end{cases}$

i.e. $\begin{cases} \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ x^2+y^2+z^2 = 1 \end{cases}$

$x \neq 0$, $y \neq 0$, $z \neq 0$.

Divide first two equations:

$\frac{y}{x} = \frac{x}{y} \Rightarrow y^2 = x^2$.

So by symmetry $x^2 = y^2 = z^2$.

So $3x^2 = 1$. So $\boxed{x = y = z = \frac{1}{\sqrt{3}}}$

So $V = 8\left(\frac{1}{\sqrt{3}}\right)^3 = \frac{8}{3\sqrt{3}} = \frac{8\sqrt{3}}{9}$

(42) Fit the plane
$z = Ax + By + C$ to the points
$\vec{r}_k = (x_k, y_k, z_k)$, where
$\vec{r}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\vec{r}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\vec{r}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{r}_4 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$,
by finding the values of A, B, C that minimize

$f(A,B,C) = \sum_{k=1}^{4} (Ax_k + By_k + C - z_k)^2$,

the sum of the squares of the deviations.

Solution

$f = C^2 + (B+C-1)^2 + (A+B+C-1)^2 + (A+C+1)^2$

$\begin{cases} f_A = 2(A+B+C-1) + 2(A+C+1) \\ f_B = 2(B+C-1) + 2(A+B+C-1) \\ f_C = 2(B+C-1) + 2(A+B+C-1) + 2(A+C+1) + 2C \end{cases}$

Need

$\frac{1}{2} f_A = 2A + B + 2C = 0$

$\frac{1}{2} f_B = A + 2B + 2C = 2$

$\frac{1}{2} f_C = 2A + 2B + 4C = 1$

Solve by Gaussian elimination or Cramer's rule:

$A = \frac{-1}{2}$, $B = \frac{3}{2}$, $C = \frac{-1}{4}$

(43) $f = x^2y^2z^2$, $g = x^2+y^2+z^2 = r^2$, $x,y,z \geq 0$.

(a) $\nabla f = \lambda \nabla g$: $\begin{aligned} 2xy^2z^2 &= \lambda 2x \\ 2yx^2z^2 &= \lambda 2y \\ 2zx^2y^2 &= \lambda 2z, \end{aligned}$

i.e. $x^2y^2z^2 = \lambda x^2 = \lambda y^2 = \lambda z^2$.

So $x = y = z = \frac{1}{\sqrt{3}} r$.

So $f_{max} = \left(\frac{r^2}{3}\right)^3$

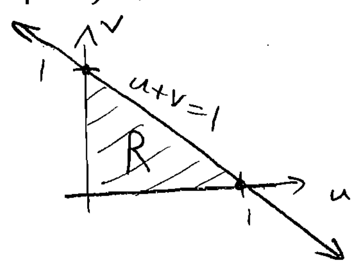
(b) Know $x^2y^2z^2 \leq \left(\frac{x^2+y^2+z^2}{3}\right)^3 \quad \forall x,y,z$.

Let $a = x^2$, $b = y^2$, $c = z^2$.

Then $abc \leq \left(\frac{a+b+c}{3}\right)^3$, i.e.

Section 15.1

(15) $f(u,v) := v - \sqrt{u}$



$$I = \iint_R f(u,v)\,du\,dv$$

$$= \int_{u=0}^{1} \int_{v=0}^{1-u} v - \sqrt{u}\,dv\,du$$

$$= \int_{u=0}^{1} \left[\frac{v^2}{2} - \sqrt{u}\,v\right]_{v=0}^{1-u} du$$

$$= \int_{u=0}^{1} \frac{(1-u)^2}{2} - \sqrt{u}\,(1-u)\,du$$

$$= \int_{u=0}^{1} \frac{1-2u+u^2}{2} - u^{\frac{1}{2}} + u^{3/2}\,du$$

$$= \left[\frac{u-u^2}{2} + \frac{u^3}{6} - \frac{2}{3}u^{3/2} + \frac{2}{5}u^{\frac{5}{2}}\right]_{u=0}^{1}$$

$$= \left[\frac{1}{6} - \frac{2}{3} + \frac{2}{5}\right]$$

$$= \frac{5 - 20 + 12}{30}$$

$$= \frac{-3}{30}$$

$$= \boxed{\frac{-1}{10}}$$

OR

$$I = \int_{v=0}^{1} \int_{u=0}^{1-v} v - \sqrt{u}\,du\,dv$$

$$= \int_{v=0}^{1} \left[vu - \frac{2}{3}u^{3/2}\right]_{u=0}^{1-v} dv$$

$$= \int_{v=0}^{1} v(1-v) - \frac{2}{3}(1-v)^{3/2}\,dv$$

[Let $w = 1-v$, so $dw = -dv$

$$= -\int_{w=1}^{0} (1-w)\,w - \frac{2}{3}w^{3/2}\,dw$$

$$= \int_{w=0}^{1} w - w^2 - \frac{2}{3}w^{3/2}\,dw$$

$$= \left[\frac{w^2}{2} - \frac{w^3}{3} - \frac{2}{3}\cdot\frac{2}{5}w^{5/2}\right]_{w=0}^{1}$$

$$= \left[\frac{1}{2} - \frac{1}{3} - \frac{4}{15}\right]$$

$$= \frac{15 - 10 - 8}{30}$$

$$= \frac{-3}{30} = \boxed{\frac{-1}{10}}$$

OR

$$I = \int_{v=0}^{1} \int_{u=0}^{1-v} v\,du\,dv - \int_{u=0}^{1} \int_{v=0}^{1-u} \sqrt{u}\,dv\,du$$

$$= \int_{v=0}^{1} \left[uv\right]_{u=0}^{1-v} dv - \int_{u=0}^{1} \left[\sqrt{u}\,v\right]_{v=0}^{1-u} du$$

$$= \int_0^1 v(1-v)\,dv - \int_0^1 \sqrt{u}\,(1-u)\,du$$

$$= \left[\frac{v^2}{2} - \frac{v^3}{3}\right]_0^1 - \left[\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} - \frac{2}{3} + \frac{2}{5}$$

$$= \frac{2}{5} - \frac{1}{2} = \frac{4}{10} - \frac{5}{10} = \boxed{\frac{-1}{10}}$$