HW 7, due Tues Oct 21

§14.3

(76) f_x, f_y continuous on R

$\Rightarrow f$ is differentiable on R (Cor thm 3)

$\Rightarrow f$ is continuous on R. (thm 4)

Note: $f(x,y) = \begin{cases} 0 & \text{if } x = 0 = y \\ \frac{xy}{x^2+y^2} & \text{otherwise} \end{cases}$ is discont. but has partials everywhere.

§14.5

(2) $f(x,y) = \ln(x^2+y^2)$.

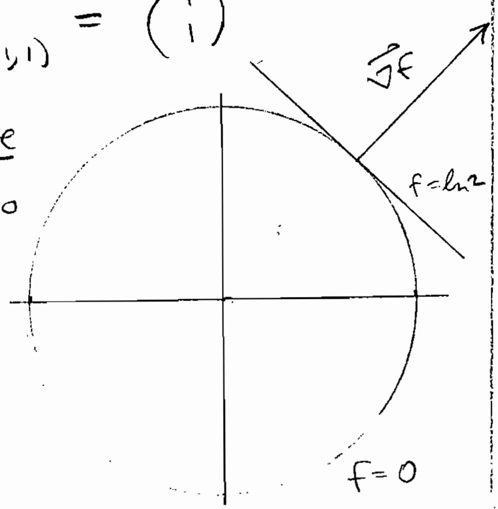
Find $\nabla f(P_0)$, where $P_0 = (1,1)$.

Sol $\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \frac{2}{x^2+y^2}\begin{pmatrix} x \\ y \end{pmatrix}$

(2) $\nabla f\big|_{(1,1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

level curve

$(x-1)+(y-1)=0$



(20) $g(x,y,z) = xe^y + z^2$.

Find $\hat{u}$ and $|\vec{u}|$ where

$\vec{u} = \nabla g\big|_{P_0}$ and $P_0 = (1, \ln 2, \frac{1}{2})$.

Sol $\nabla g = \begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix} = \begin{pmatrix} e^y \\ xe^y \\ 2z \end{pmatrix}$.

$\vec{u} = \nabla g\big|_{P_0} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

$\hat{u} = \boxed{\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}$, $-\hat{u} = \boxed{-\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}$

$D_{\hat{u}} \nabla g = |\nabla g| = \boxed{3}$

$D_{-\hat{u}} \nabla g = -|\nabla g| = \boxed{-3}$

§14.6

(18) Find parametric equations for the line tangent to the curve of intersection of the surfaces

$f(x,y,z) := x^2+y^2 = 4$,

$g(x,y,z) := x^2+y^2-z = 0$

at the point $(\sqrt{2}, \sqrt{2}, 4) =: \vec{r}_0$

Solution

The line tangent to the curve of intersection lies in the intersection of the tangent planes: and so has direction vector parallel to $\nabla f \times \nabla g$.

$\nabla f = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}$ and $\nabla g = \begin{pmatrix} 2x \\ 2y \\ -1 \end{pmatrix}$,

So at point $\vec{r}_0$,

$\nabla f = 2\sqrt{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\nabla g = \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \\ -1 \end{pmatrix}$.

So $\nabla f \times \nabla g = 2\sqrt{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

So $\boxed{\vec{r}(t) = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 4 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$.

(22) $h(x,y,z) = \cos(\pi xy) + xz^2$,

$\vec{r}_0 = (-1,-1,-1)$, $\boxed{dr = \frac{0.1}{\sqrt{3}}(1,1,1)}$

$dh = dr \cdot \nabla h$. $\nabla h = \begin{pmatrix} -\sin(\pi xy)\pi y + z^2 \\ -\sin(\pi xy)\pi x \\ 2xz \end{pmatrix}$

$\nabla h\big|_{\vec{r}_0} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

$dh = \frac{(.1)}{\sqrt{3}}(1+0+2) = \frac{.3}{\sqrt{3}} = \boxed{\frac{\sqrt{3}}{10} \simeq .1732}$

(34) $f(x,y) = xy^2 + y\cos(x-1)$ near $P_0 = (1,2)$.

$f(x_0,y_0) = 4 + 2 = 6$.

$\nabla f = \begin{pmatrix} y^2 - y\sin(x-1) \\ 2xy + \cos(x-1) \end{pmatrix}$. $\nabla f\big|_{P_0} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

$\boxed{L(x,y) = 6 + 4(x-1) + 5(y-2)} + E(x,y)$ $\boxed{4x+5y-8}$

$f_{xx} = -y\cos(x-1)$, $f_{xy} = 2y - \sin(x-1)$

$f_{yy} = 2x$. $x \in [.9, 1.1]$, $y \in [1.9, 2.1]$

$M \le 4.3$

$E \le \frac{1}{2}M(|x-x_0| \cdot |y-y_0|)^2 \le \frac{1}{2}(4.3)\cdot(.04) = \boxed{.086}$

§14.6 (HW 7)

22 Given

$h(x,y,z) = \cos(\pi xy) + xz^2$,

$\vec{P}_0 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

Find dh if $\vec{P}$ moves from P_0 a distance $ds = .1$ toward $\vec{O}$.

Solution

$\hat{u} \parallel (\vec{O} - \vec{P}_0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\hat{u} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$d\vec{P} = ds \cdot \hat{u} = \frac{.1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$(dh = d\vec{P} \cdot \nabla h = ds \cdot \hat{u} \cdot \nabla h.)$

$\nabla h = \begin{pmatrix} -\sin(\pi xy)\pi y + z^2 \\ -\sin(\pi xy)\pi x \\ 2xz \end{pmatrix}$

$\nabla h \Big|_{P_0} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$dh = d\vec{P} \cdot \nabla h = (ds)\, \hat{u} \cdot \nabla h$

$= (.1) \left[\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$= (.1) \frac{3}{\sqrt{3}} = \frac{\sqrt{3}}{10} \simeq .1732$

$\boxed{\S 14.3}$ HW 6, due Tues Oct 14

(28) $f(x,y,z) = \sec^{-1}(x+yz)$

Recall $(\sec^{-1})'(u) = \dfrac{1}{|u|\sqrt{u^2-1}}$.

Indeed:

$\dfrac{d}{du}\left[\sec(\sec^{-1}(u)) = u\right]$

By chain rule, get:

$(\sec^{-1})'(u) \cdot \sec'(\sec^{-1}(u)) = 1$

i.e. $(\sec^{-1})'(u) = \dfrac{1}{\sec'(\sec^{-1}(u))}$.

But $\sec'(\sec^{-1}(u))$

$= \sec(\sec^{-1}(u)) \cdot \tan(\sec^{-1}(u))$

$= u \cdot \left[\mathrm{sgn}(u) \cdot \sqrt{\sec^2(\sec^{-1}(u)) - 1}\right]$

$= |u| \cdot \sqrt{u^2-1}$.

So:

$$\nabla f = \left(\frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}\right) \cdot \begin{pmatrix} 1 \\ z \\ y \end{pmatrix}$$

(34) $f(x,y,z) = \sinh(xy - z^2)$.

Recall $\sinh(u) =$ odd part of e^u

$= \dfrac{e^u - e^{-u}}{2}$,

$\cosh(u) =$ even part of e^u

$= \dfrac{e^u + e^{-u}}{2}$, and

$\sinh'(u) = \cosh(u)$,

$$\nabla f = \cosh(xy - z^2) \cdot \begin{pmatrix} y \\ x \\ -2z \end{pmatrix}.$$

(44) $h(x,y) = xe^y + y + 1$

$h_x = e^y \qquad h_y = xe^y + 1$

$h_{xx} = 0 \qquad h_{yy} = xe^y$

$h_{xy} = e^y = h_{yx}$ ✓

$\boxed{\S 14.6}$

(6) $\underbrace{x^2 - xy - y^2 - z}_{\text{Call } f} = 0$,

$P_0(1,1,-1)$

$$\nabla f = \begin{pmatrix} 2x - y \\ -x - 2y \\ -1 \end{pmatrix}$$

$$\nabla f\big|_{P_0} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} =: \vec{n}$$

$f|_{P_0} = 0$ ✓

equation of tangent line is

$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$, i.e.

$$\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-1 \\ z+1 \end{pmatrix} = 0, \text{ i.e.}$$

(a) $\boxed{x - 3y - z = -1}$

(b) normal line: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$

(8) $f(x,y,z) = x^2 + y^2 - 2xy - x + 3y - z = -4$,

$P_0 = \begin{pmatrix} 2 \\ -3 \\ 18 \end{pmatrix}$.

$$\nabla f = \begin{pmatrix} 2x - 2y - 1 \\ 2y - 2x + 3 \\ -1 \end{pmatrix} \Rightarrow \nabla f(P_0) = \begin{pmatrix} 9 \\ -7 \\ -1 \end{pmatrix}$$

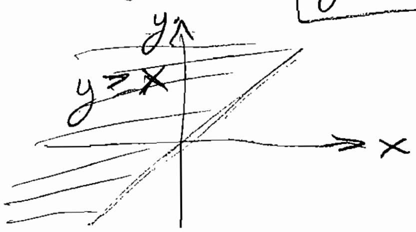
(a) $\begin{pmatrix} 9 \\ -7 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x-2 \\ y+3 \\ z-18 \end{pmatrix} = 0 \Rightarrow \boxed{9x - 7y - z = 21}$

(b) $x = 2 + 9t$
$y = -3 - 7t$
$z = 18 - t$

§14.1 HW 5 due Sept 7

(2) $f(x,y) = \sqrt{y-x}$

(a) $\text{dom} f = \{(x,y) : y - x \geq 0,$ i.e. $\boxed{y \geq x}\}$



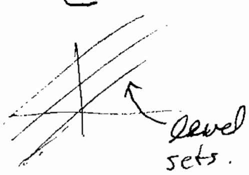
(b) range $f = \{z : \boxed{z \geq 0}\}$

(c) level curves:

$f(x,y) = \sqrt{y-x} = c$

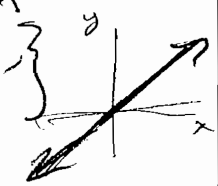
$\Rightarrow \; y - x = c^2$

$y = x + c^2$



(d) boundary of domain of f

$= \{(x,y) : y - x = 0,$ i.e. $y = x\}$



(e) domain is <u>closed</u>, since it includes its boundary.

(f) domain is <u>unbounded</u>, since it has points an arbitrary distance from the origin.

(46) $f(x,y,z) = xy - z$

$x = t - 1$

$y = t - 2$

$z = t + 7$

$g(t) = f(x(t), y(t), z(t)) = (t-1)(t-2) - (t+7)$

$g'(t) = (t-1) + (t-2) - 1$

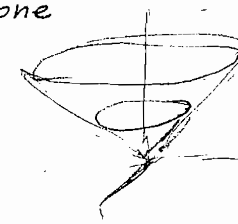
$= 2t - 4$

Set $g'(t) = 0$. Then $t = 2$

and $\boxed{g(t) = -9}$ ← minimum value, since $g(t)$ is concave up

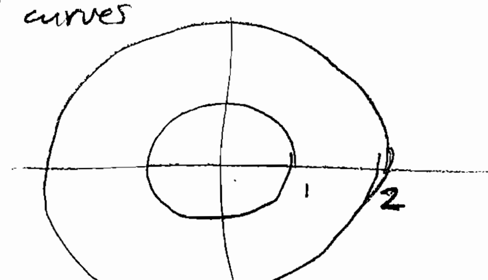
(22) $z = f(x,y) = \sqrt{x^2+y^2}$

(a) z is the radial distance from the origin. So $z = f(x,y)$ is the equation of the top nappe of a cone



cone, not paraboloid x^2+y^2

(b) level curves



§14.2

(36) $f(x,y) = \dfrac{x^4}{x^4+y^2}$

$\lim_{x\to 0} f(x,kx) = \dfrac{x^4}{x^4+k^2x^2} = 1.$

$\lim_{x\to 0} f(x,x^2) = \dfrac{x^4}{x^4+x^4} = \dfrac{1}{2}$

(28) $f(x,y) = 1 - |x| - |y|$

$1 - |x| - |y| = c$

$|x| + |y| = \cancel{1}-c \geq 0 \Rightarrow c \leq 1$