Answers to Practice Final

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(1) (i) length = 4

(ii) 14/3

(iii) 74/9

(2) (a) $-3/\sqrt{6}$

(b) $\frac{1}{\sqrt{14}}(2, 1, -3)$

(c) any unit vector $\perp$ to $(2, 1, -3)$, e.g., $(1/\sqrt{5})(-1, 2, 0)$

(d) $|D_{\hat{\mathbf{u}}}| \le |\nabla f| = \sqrt{14} < |-20|$, so $D_{\hat{\mathbf{u}}} f \neq -20$ for all $\hat{\mathbf{u}}$.

(e) -12

(3) (a) $3x + 5z = 10$

(b) $(z_x, z_y) = -(f_x, f_y)/f_z = (-3/5, 0)$

(c) no local extremum of z can occur at $(0, 0)$, because $\nabla z(0, 0)$ exists and is nonzero.

(4) (i) $\frac{\partial M}{\partial y} = 3x^2 e^{x^3 y} + 3yx^5 e^{x^3 y} = \frac{\partial N}{\partial x}$

(ii) $f = e^{x^3 y}$

(iii) $e^1 - 1$

(5) (i) $\int_C \mathbf{F} \cdot \mathbf{n} = \oint_C M\,dy - N\,dx = \iint_D \frac{\partial M}{dx} + \frac{\partial N}{dy} = \iint_D \nabla \cdot \mathbf{F}$

(ii) 24π

(iii) $f = \frac{x^4}{4} + \frac{y^4}{4}$, and the integral around a closed loop of the gradient of a potential is zero.

(6) (i) $a/2 \le \rho \le a$,
$0 \le \theta \le \pi/2$,
$\pi/6 \le \phi \le \pi/4$.

(ii) $\frac{15\pi(\sqrt{3} - \sqrt{2})a^4}{2^8}$

(7) (i) $|\mathbf{r}'| = 3$, length $= 2\pi\sqrt{3}$,

$$\mathbf{R}(s) = \mathbf{r}(t(s)) = \begin{pmatrix} \cos(s/\sqrt{3}) - \sin(s/\sqrt{3}) \\ \cos(s/\sqrt{3}) + \sin(s/\sqrt{3}) \\ s/\sqrt{3} \end{pmatrix},$$

$s \in [0, 2\pi\sqrt{3}$, where $t(s) = s/\sqrt{3}$.

(ii)

* $$\hat{\mathbf{T}} = \mathbf{R}'(s) = \frac{1}{\sqrt{3}} \begin{pmatrix} -\sin(s/\sqrt{3}) - \cos(s/\sqrt{3}) \\ -\sin(s/\sqrt{3}) + \cos(s/\sqrt{3}) \\ 1 \end{pmatrix}.$$
* $\kappa\hat{\mathbf{N}} = \frac{d\hat{\mathbf{T}}}{ds}$ gives
$$\hat{\mathbf{N}} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\cos(s/\sqrt{3}) + \sin(s/\sqrt{3}) \\ -\cos(s/\sqrt{3}) - \sin(s/\sqrt{3}) \\ 0 \end{pmatrix}$$
and $\kappa = \sqrt{2}/3$.
* $$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}} = \frac{1}{\sqrt{2}\sqrt{3}} \begin{pmatrix} \cos(s/\sqrt{3}) + \sin(s/\sqrt{3}) \\ -\cos(s/\sqrt{3}) + \sin(s/\sqrt{3}) \\ 2 \end{pmatrix}.$$

(iii)

* $\kappa = \frac{\sqrt{2}}{3}$.
* $\tau = \frac{-1}{3}$.

References

[1] http://www.math.wisc.edu/~seeger/234/fpractice.pdf