MATH 234 (A. Assadi) Fall 2009 Final Exam ([redacted])

Instructions. (1) This test is CLOSED BOOK. You are allowed to have two index-cards or one page of notebook paper with formulas only. Calculators are allowed ONLY for computation and graphing.

(2) The time is 120 minutes for working on the test.

(3) All functions and vector fields are assumed to be differentiable everywhere, unless otherwise stated.

We rely on honor system for the students to use only their own work and without asking from or offering help to other students. Please Observe the Honor System.

Time: 120 minutes

YOUR NAME: **TA NAME:**

SECTION:

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
8	15	
Total	120	

Problem 1.

(a) Compute the gradient of the function

$$g(x, y, z) = e^{x+y} + \cos(x - y) + z^2.$$

$$\nabla g = (e^{x+y} - \sin(x-y))\hat{\imath} + (e^{x+y} + \sin(x-y))\hat{\jmath} + (2z)\hat{k}$$

(b) Compute curl($\mathbf{F} + \mathbf{G}$) where

$$\mathbf{F} = \left(-\frac{1}{2}\sin(x + y), \frac{1}{2}\sin(x + y), z\right),$$
$$\mathbf{G} = (x + y, x - y, z^2).$$

$$\text{curl } F = \cos(x+y)\hat{k}$$

$$\text{curl } G = \vec{0}$$

$$\text{curl}(F+G) = \cos(x+y)\hat{k}$$

(c) Let $\mathbf{H} = \sin y\hat{\mathbf{i}} + e^x y\hat{\mathbf{j}} + (z^2 + 1)\hat{\mathbf{k}}$.
Compute div($\mathbf{H}$ + curl($\nabla g + \mathbf{G}$)).

$$= \text{div } H$$
$$= e^x + 2z$$

Problem 2. Consider the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 4$.

(a) Compute the surface area of the part of the sphere between the planes $z = 0$ and $z = 1$.

Area $= 4\pi$

(b) Compute the surface area of the part of the cylinder between the same two planes.

Area $= 4\pi$

Problem 3. Suppose A is the region between two squares of side 4 and side 2 that have the same center at the origin. Let $\mathbf{F} = M\hat{\imath} + N\hat{\jmath}$, where $M = y(1 - xy)$ and $N = x(1 + xy)$.

(a) Use Green's theorem to write the circulation $\oint_{\partial A} M\,dx + N\,dy$ as an iterated double integral, and identify the integration bounds.

$$\int_{-2}^{2}\int_{-2}^{2} 4xy\,dx\,dy - \int_{-1}^{1}\int_{-1}^{1} 4xy\,dx\,dy$$

(b) Find $\oint_{\partial A} M\,dx + N\,dy$.

$$= 0$$

Problem 4. The plane $4x - y + z = 2$ cuts the cone $x^2 - y^2 - z^2 = 0$ in an ellipse. Using the method of Lagrange multipliers, find the greatest and smallest values hat the function $f(x, y, z) = x^2$ takes on the ellipse. Write the ~~two~~ equations that you need to solve and circle them. Put your final answers on the lines at the bottom.

$$\boxed{\begin{aligned} 2x &= \lambda 4 + \mu 2x \\ 0 &= -\lambda - \mu 2y \\ 0 &= \lambda - \mu 2z \\ 4x - y + z &= 2 \\ x^2 - y^2 - z^2 &= 0 \end{aligned}}$$

$\Rightarrow z = -y.$

Get system of two equations

$$\boxed{\begin{aligned} 4x - 2y &= 2 \\ x^2 - 2y^2 &= 0 \end{aligned}}$$

greatest value: $\dfrac{2}{(2\sqrt{2}-1)^2}$ smallest value: $\dfrac{2}{(2\sqrt{2}+1)^2}$

Problem 5. Compute the outward flux of $\mathbf{G} = (x + 3y)\hat{\mathbf{\imath}} + (2 - 4z)\hat{\mathbf{k}}$ across the boundary of the lower half of the solid ~~unit~~ sphere, that is, across the boundary of the set

$$D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 9, \quad z \leq 0\}.$$

$$\iiint_{\partial D} \hat{n} \cdot G = \iiint_{D} \nabla \cdot G$$

$$= -2\pi \cdot 3^2$$

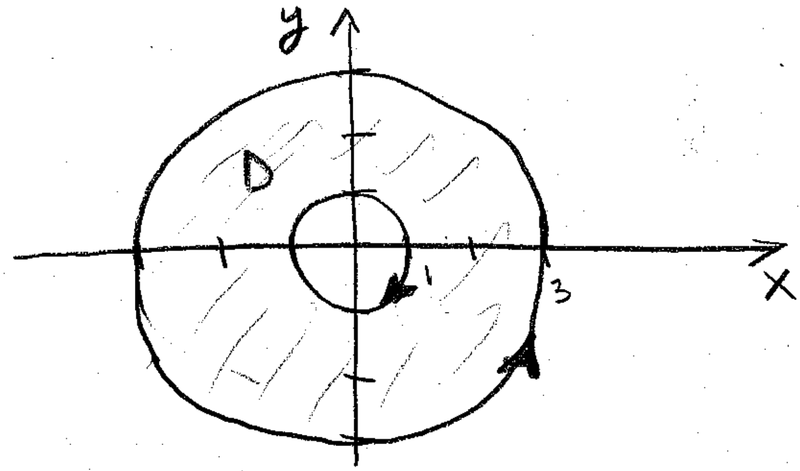
$$= -18\pi$$

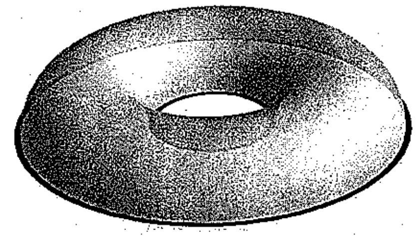
Problem 6. Consider the surface S, which is the upper half of the torus (see figure) and whose parametrization is

$$\mathbf{r}(u,v) = \langle (2+\cos u)\cos v,\ (2+\cos u)\sin v,\ \sin u\rangle, \quad 0 \le u \le \pi, \quad 0 \le v \le 2\pi.$$

Consider also the annual ring in the (xy)-plane that has the same boundary as S.

(a) Identify and sketch the boundary of the surface.





(b) Let $\mathbf{F} = (y)\hat{\mathbf{\imath}} - (x)\hat{\mathbf{\jmath}} + (e^{\sin^{10000} z})\hat{\mathbf{k}}$. Compute $\iint_S \hat{\mathbf{n}}\cdot \operatorname{curl}\mathbf{F}\, d\sigma$, where $\hat{\mathbf{n}}$ is the unit normal in the direction with positive z component.

$$\iint_S \hat{n}\cdot \nabla\times F$$

$$= \oint_{\partial S} \vec{F}\cdot d\vec{r}$$

$$= \iint_{\left\{ \substack{z=0 \\ 1\le x^2+y^2\le 3^2} \right\} =: D} \hat{k}\cdot \nabla\times F$$

$$= \iint_D (-2)$$

$$= \boxed{-16\pi}$$

Problem 7.
One of the following two vector fields is conservative and the other is not:

$$\mathbf{F}_1(x, y, z) = (y - z)\hat{\mathbf{i}} + (x - z)\hat{\mathbf{j}} + (x - y)\hat{\mathbf{k}}.$$
$$\mathbf{F}_2(x, y, z) = (y + z)\hat{\mathbf{i}} + (x + z)\hat{\mathbf{j}} + (x + y)\hat{\mathbf{k}},$$

(a) Which vector field is conservative? Which one *is not* conservative?

$$\nabla \times \vec{F}_1 = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \Rightarrow \boxed{\vec{F}_1 \text{ is } \underline{\text{not}} \text{ conservative}}$$

$$\nabla \times \vec{F}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{\vec{F}_2 \text{ is conservative.}}$$

(b) For the vector field $\mathbf{F}$ that you found to be conservative, find a potential function ϕ for which $\mathbf{F} = \nabla\phi$.

$$\varphi = xy + xz + yz + C_0$$

(c) For the vector field $\mathbf{F}$ that you found to be conservative, *use the fact that it is conservative* to evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is the oriented curve parametrized by $\mathbf{r}(t) = (\sin t \cos t, \cos t, \sin t)$, $0 \leq t \leq 2\pi$.

Integral of a conservative vector field around a closed loop is $\boxed{\text{zero}}$.

Problem 8. Calculate the work of the vector field $\mathbf{F} = (xy+1)\hat{\mathbf{i}} + (y^2+1)\hat{\mathbf{j}}$ from $(1,0)$ to $(-1,0)$ along the curve $x^2+y^2=1,\ \ y \geq 0$.

Work $= \boxed{-2}$