MATH 234 (A. Assadi)

Spring 2009

MIDTERM I Thursday, March 12, 2009

NAME (print): Alec Johnson

Instructions:

1. Write your name on each page.
   Circle the name of your TA:

   Alec   Fang   George

Closed book. Closed notes. Calculators allowed. No laptops.

2 index cards of size 4" x 6" with formulas allowed.

Answer your questions on the exam paper: There are some extra pages in the back if you need them.

Show all your work. Partial credit is given only if your work is clear.

Enclose your final answers clearly in a rectangle.

Time allowed: 70 minutes for your work, plus 5 minutes for checking your answers.

Problem 1	20	
Problem 2	20	
Problem 3	20	
Problem 4	20	
Problem 5	20	
Problem 6	20	
Problem 7	20	
Total	140	

Problem 1. A particle is moving downwards along the helix with position **F**(t), velocity vector **V**, and acceleration vector **A**.

$$\mathbf{F}(t) = 2\sin(3t)\,\mathbf{i} + 2\cos(3t)\,\mathbf{j} - (6t)\,\mathbf{k}.$$

(a) Compute the velocity **V**, the speed S and the acceleration **A** of the particle.

(b) How far does the particle travel along its path from time t = T to t = T + 2π? Show your work and explain it, simplify as much as you can, but you do not need to compute the final integrals. Enclose your final answer in a box.

(c) Find the *unit tangent* vector **T** and the *unit normal* vector **N** for the trajectory of the particle.

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(a) $\vec{V} = \vec{F}'(t)$

(3) $= \begin{pmatrix} 6\cos 3t \\ -6\sin 3t \\ -6 \end{pmatrix}$

$S = |V|$

$= \sqrt{6^2\cos^2(3t) + 6^2\sin^2 3t + 6^2}$

$= \sqrt{2\cdot 6^2}$

(3) $\boxed{S = 6\sqrt{2}}$

$\vec{A} = \vec{F}''(t)$

(3) $= \begin{pmatrix} -18\sin 3t \\ -18\cos 3t \\ 0 \end{pmatrix}$

(b) $\text{Length} = \int_T^{T+2\pi} ds$

$= \int_T^{T+2\pi} |v|\,dt$

(3) $= \boxed{\int_T^{T+2\pi} 6\sqrt{2}}$

$\left(= 2\pi\cdot 6\sqrt{2} = \boxed{12\sqrt{2}\,\pi} \right)$

(3) (c) $\hat{T} = \dfrac{\vec{V}}{|\vec{V}|} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} \cos 3t \\ -\sin 3t \\ -1 \end{pmatrix}$

(2)

$\vec{A}\cdot\hat{T} = 0.$ So $\vec{A}_{\perp} = \vec{A}$

(3)

(5) So $\hat{N} = \dfrac{\vec{A}_{\perp}}{|A_{\perp}|} = \dfrac{\vec{A}}{|A|} = \begin{pmatrix} -\sin 3t \\ -\cos 3t \\ 0 \end{pmatrix}$

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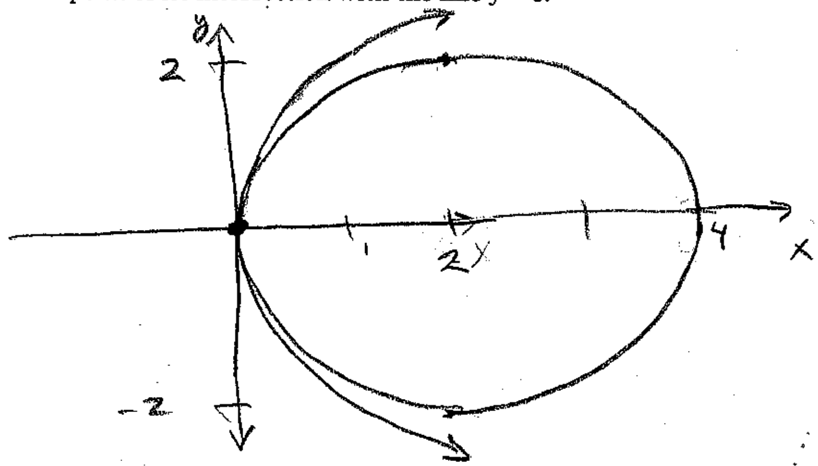
Problem 2. Find and graph the circle of curvature for the parabola $4x=y^2$ at the point of its intersection with the line $y = 0$.

(4)



Parametrization

(10) (6) $\vec{r}(t) = \begin{pmatrix} \frac{1}{4}t^2 \\ t \end{pmatrix}$ (2)

$r' = \begin{pmatrix} \frac{1}{2}t \\ 1 \end{pmatrix}$ $\quad r'\big|_{t=0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (2)

$r'' = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$ $\quad r''\big|_{t=0} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$ (2)

(2) $K = \dfrac{|r' \times r''|}{|r'|^3}$ (1)

$= \dfrac{\frac{1}{2}}{1}$

$\boxed{K = \frac{1}{2}}$

(2) $R = \frac{1}{K} = 2.$

(6)

(2) center: $(2, 0)$

equation of circle:

(4) $\boxed{(x-2)^2 + y^2 = 4}$

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Problem 3. A function $z = f(x,y)$ is given by the equation:

$$z^3 = 3(x^3 + y^3).$$

Find the gradient vector of f at P= (2 , 1).

$$z^3 = 3(x^3+y^3).$$

$$\cancel{3}z^2\,dz = \cancel{3}(3x^2dx + 3y^2dy)$$

$$z^3\big|_p = 3(1+8) = 27$$

$$z\big|_p = 3$$

linearization:

$$9\,dz = 12\,dx + 3\,dy,$$

i.e., $3dz = 4\,dx + dy$

So
$$\boxed{\begin{aligned} \frac{\partial z}{\partial x}\Big|_p &= \frac{4}{3}, \\ \frac{\partial z}{\partial y}\Big|_p &= \frac{1}{3} \end{aligned}}$$

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Problem 4. Find the extreme values of $f(x, y) = 6x^2 + 2y^2 + 4x$ on the circle centered at the origin and of radius 2.

$$f = 6x^2 + 2y^2 + 4x$$

$$g = x^2 + y^2 = 4$$

Set $\nabla f = \lambda \nabla g$.

$$\begin{cases} 12x + 4 = \lambda 2x \\ 4y = \lambda 2y \\ x^2 + y^2 = 4 \end{cases}$$

Cross-multiply:

$$\lambda \cancel{2} y (12x+4) = \lambda \cancel{2} x \cdot 4y$$

$$\lambda y (8x+4) = 0$$

Cases

$y = 0$:

$x = \pm 2$,

Can solve for λ.

$\lambda = 0$:

$\Rightarrow y = 0$,

probably a contradiction but in any case no new information.

$8x + 4 = 0$:

$x = \frac{-1}{2}$

$y^2 = 3 + \frac{3}{4} = \frac{15}{4}$

$\lambda = 2$

Test candidates

x	y	f
-2	0	16
2	0	32 (max)
$-\frac{1}{2}$	$\frac{\pm\sqrt{15}}{2}$	$\frac{3}{2} + \frac{15}{2} - 2 = 7$ (min)

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Problem 5. Let

$$f(x, y) = 9x^2 + 16y^2 + 1.$$

(a) Sketch the level sets f (x, y) = c for c = 0, 2, 26.

(b) Sketch the graph of intersection of the (x,z)-plane and the surface given by z = f (x, y) and call the curve C. Find all points on C that have a tangent parallel to any of the coordinate axes, and call one such point P.

(c) Find a *unit* vector v, in the direction in which *f(x,y) changes* most rapidly at the point P from part (b). What is the rate of change in this direction?

(a) Let $h = f - 1$.

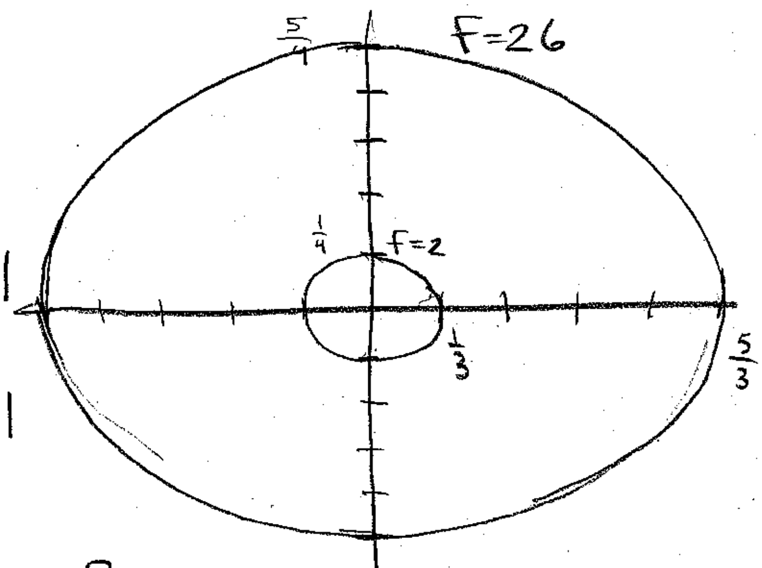
$f = 0, 2, 26$ when

$h = -1, 1, 25$.

$h = -1$: no solution.

$h = 1$: $\left(\dfrac{x}{\left(\frac{1}{3}\right)}\right)^2 + \left(\dfrac{y}{\left(\frac{1}{4}\right)}\right)^2 = 1$

$h = 25$: $\left(\dfrac{x}{\left(\frac{5}{3}\right)}\right)^2 + \left(\dfrac{y}{\left(\frac{5}{4}\right)}\right)^2 = 1$



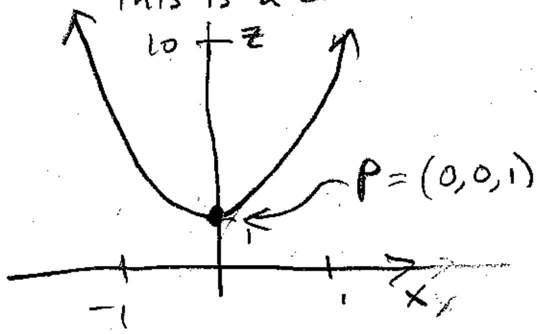
(b) x-z plane has equation $y = 0$.

So seek intersection of:

$$\begin{cases} z = 9x^2 + 16y^2 + 1 \\ y = 0. \end{cases}$$

i.e. $\begin{cases} z = 9x^2 + 1, \\ y = 0 \end{cases}$

This is a curve in the x-z plane:



(c) At $P = (0,0,1)$

f changes most rapidly in the direction of ∇f.

$\nabla f \Big| = \begin{pmatrix} 18x \\ 32y \end{pmatrix}$

$\nabla f \Big|_P = 0$

In any direction the rate of change is zero.

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Problem 6.

(a) Find all points at which gradient of the function f vanishes:

$$f(x, y) = 2x^4 - 4x^2 + y^2 + 2y$$

(b) Indicate whether each point is a local maximum, local minimum or a saddle point.

(a) $\nabla f = \begin{pmatrix} 8x^3 - 8x \\ 2y + 2 \end{pmatrix}$

Set $\nabla f = 0$.

Then: $\begin{cases} 8x(x^2-1) = 0 \\ y = -1 \end{cases}$

$x \in \{0, -1, 1\}$.

(b) $f_{xx} = 24x^2 - 8 = 8(3x^2-1)$

$f_{yy} = 2$

$f_{xy} = 0$

Let $D = f_{xx} f_{yy} - f_{xy}^2$.

$D = 16(3x^2-1)$.

x	y	D		f
0	-1	-16	Saddle	-1
-1	-1	48	local min	1
1	-1	48	local min	-3

↑ (global min)

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Problem 7. Find the equation of the tangent plane and the normal line for the surface S given by

$$x^2 - 5(y^2) - 25(z^2) = -1$$

at the point p = (2, 1, 0).

p satisfies. ✓

$$\left(2x\,dx - 10y\,dy - 50z\,dz = 0\right)\Big|_p$$

$$4\,dx - 10\,dy = 0$$

$$2\,dx - 5\,dy = 0$$

$$2(x-2) - 5(y-1) = 0$$

$$\boxed{2x - 5y = -1}$$

normal vector : $\begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}$

normal line :

$$\vec{r}(t) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}$$

i.e. $x = 2 + 2t$

$y = 1 - 5t$

i.e. $t = \dfrac{x-2}{2} = \dfrac{y-1}{-5}$

i.e. $-5(x-2) = 2(y-1)$

i.e. $\boxed{2y + 5x = 8}$

"Alternative" method :

Use the fact that the gradient vector at each point of the level surface is normal to the surface and its tangent plane at that point.

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