Solutions for Practice Exam for Midterm II

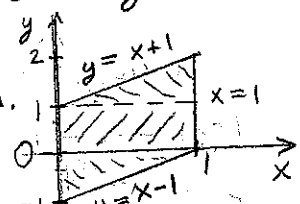
(1)
$$I := \int_{-1}^{1} dx \int_{-x}^{x^2} (3+2y)\,dy$$
$$= \int_{-1}^{1} dx \left[3y + y^2\right]_{y=-x}^{x^2}$$
$$= \int_{-1}^{1} 3x^2 + \overset{\text{odd}}{\cancel{3x}} + x^4 - x^2 \,dx$$
$$= 2\int_{0}^{1} 2x^2 + x^4 \,dx$$
$$= 2\left[\frac{2x^3}{3} + \frac{x^5}{5}\right]_{x=0}^{1}$$
$$= 2\left(\frac{2}{3} + \frac{1}{5}\right)$$
$$= \frac{26}{15}$$

(2)
$$I := \int_0^1 dx \int_{x-1}^{x+1} e^{y+1}\,dy.$$

Change order of integration.

Solution.

Sketch region.

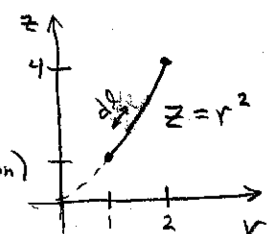


$$I = \int_{-1}^{0} dy \int_{0}^{y+1} e^{y+1}\,dx + \int_{0}^{1} dy \int_{0}^{1} e^{y+1}\,dx + \int_{1}^{2} dy \int_{y-1}^{1} e^{y+1}\,dx$$

(3) Find the area of the surface
$$z = x^2 + y^2$$
between the planes
$$z = 1,\quad z = 4.$$

Solution.

Sketch: (surface of revolution)



$$A = \iint (r\,d\theta)\,dl$$
$$= \int_{\theta=0}^{2\pi} \int_{r=1}^{2} r\left(\frac{dl}{dr}\right) dr\,d\theta$$

(3 cont.)

$dl^2 = dr^2 + dz^2$, so
$$dl = \sqrt{1+\left(\frac{dz}{dr}\right)^2}\,dr$$
$$A = \int_{\theta=0}^{2\pi} d\theta \int_{r=1}^{2} \sqrt{1+\left(\frac{dz}{dr}\right)^2}\; r\,dr$$
$$= 2\pi \int_{r=1}^{2} \sqrt{1+4r^2}\; r\,dr$$
$$= 2\pi \left[\frac{\frac{2}{3}(1+4r^2)^{3/2}}{8}\right]_{r=1}^{2}$$
$$= \frac{\pi}{6}\left[17^{3/2} - 5^{3/2}\right].$$

(4) A spherical shell S is made of two spheres with center at origin, with the inner radius $R_1 = \sqrt{2}$ and the outer radius $R_2 = 2$.

A solid V is the volume cut off from S by a cone K. The cone K is obtained from a 360 degree rotation about the z-axis of the triangle with vertices $\begin{pmatrix}0\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\2\end{pmatrix}$, and $\begin{pmatrix}0\\2\\2\end{pmatrix}$

(a) Graph the intersection of S, K, and the volume V with the three coordinate planes.