§1.6 #52

$y^3 y'' = 1.$

This belongs to the family $F(y, y', y'') = 0.$

Use the substitution $p = y'$ and treat y as the <u>dependent</u> variable.

$$y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy} \cdot p$$

So the ODE $y^3 y'' = 1$ becomes

$$y^3 \cdot p \frac{dp}{dy} = 1.$$ Separating,

$$\int p\, dp = \int y^{-3}\, dy$$

$$\frac{p^2}{2} = \frac{y^{-2}}{-2} + \frac{C}{2}.$$ Since $p = y'$;

$$(y')^2 = -y^{-2} + C.$$

Solve for y' and separate:

$$y' = \pm\sqrt{C - y^{-2}}.$$

Rather than consider positive and negative cases we make convenient choices to get <u>a</u> general (but not <u>the</u> general) answer. We can then consider how to generalize the solution so we can match <u>any</u> initial conditions, thus showing that we have <u>the</u> general solution.

Assume $y' = +\sqrt{C - y^{-2}}$

$y > 0$, $C - y^{-2} > 0.$

Then:

$$y' = \sqrt{C - \frac{1}{y^2}} = \sqrt{\frac{Cy^2 - 1}{y^2}} = \frac{\sqrt{Cy^2 - 1}}{y}.$$

So $$\int \frac{y}{\sqrt{Cy^2 - 1}}\, dy = \int dx$$

Let $u = Cy^2 - 1$

So $du = 2Cy\, dy.$

So $$\frac{1}{2C}\int \frac{du}{\sqrt{u}} = \int dx$$

$$\frac{\sqrt{u}}{C} = x + C_2$$

$$\sqrt{u} = (Cx + B)$$

$$u = (Cx + B)^2$$

$$\boxed{Cy^2 - 1 = (Cx + B)^2}$$

This is an implicit solution. By differentiating implicitly twice and eliminating y' we should be able to check it.

$$\cancel{2C} y y' = \cancel{2C}(Cx + B)$$

$$(y')^2 + y y'' = C.$$

But $(y')^2 = \left(\frac{Cx + B}{y}\right)^2 = \frac{Cy^2 - 1}{y^2}$, so

$$Cy^2 - 1 + y^3 y'' = Cy^2$$

i.e. $y^3 y'' = 1$, as desired.