Fluid laws

Material Derivative:

$$\frac{d\alpha}{dt} = \frac{\partial \alpha}{\partial t} + \frac{dx_i}{dt}\frac{\partial \alpha}{\partial x_i} = \boxed{\frac{\partial \alpha}{\partial t} + u_i \frac{\partial \alpha}{\partial x_i}}$$

$$d_t \alpha = \alpha_t + \underset{\sim}{u} \cdot \nabla \alpha$$

Reynold's Transport Theorem:

$$\frac{d}{dt}\int_V \alpha \, dV = \int_V \alpha_t + \nabla \cdot (\alpha \underset{\sim}{u}) \, dV$$

where ∂V moves with $\underset{\sim}{u}$.

$$= \int_V \alpha_t + \int_{\partial V} (\hat{n} \cdot \underset{\sim}{u}) \alpha$$

General Conservation Law:

$$\frac{\partial \varphi}{\partial t} + \nabla \cdot \underset{\sim}{q} = P(x,t)$$

where φ is the amount of "stuff",
q is the flux rate,
and P is the production rate.

Advection–Diffusion Constitutive law:

$$\underset{\sim}{q} = \underbrace{\varphi \underset{\sim}{u}}_{\text{macroscopic transport}} - \underbrace{K \underset{\sim}{\nabla} \varphi}_{\text{microscopic/molecular transport term}}$$

Advection–diffusion equation

$$\partial_t \varphi + \underset{\sim}{\nabla} \cdot (\varphi \underset{\sim}{u}) = \underset{\sim}{\nabla} \cdot (K \underset{\sim}{\nabla} \varphi) + P$$

Special cases:

① $K=0$, $P=0$

$$\partial_t \varphi + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \varphi + \varphi \underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0$$

+ Incompressible ($\nabla \cdot \underset{\sim}{u} = 0$)

$$\boxed{\partial_t \varphi + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \varphi = 0} \leftarrow$$ ('Transport' eqn. ≡ one-way wave eqn.)

② $\underset{\sim}{u} = 0$, $P=0$

$$\partial_t \varphi = \nabla \cdot (K \nabla \varphi)$$ (Diffusion eqn, heat eqn.)

K constant: $\partial_t = K \nabla^2 \varphi$

③ $u=0$, steady state $\equiv (\partial_t = 0)$

$$-\nabla \cdot (K \nabla \varphi) = P$$

K constant: $-\nabla^2 \varphi = \frac{P(x)}{K} \equiv f(x)$ (Poisson's eqn)

$P=0$: $\nabla^2 \varphi = 0$ (Laplace's eqn)

Conservation of mass ("continuity equation")

$$\frac{D}{Dt}\int_V \rho \, dV = 0$$

$$\boxed{\rho_t + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{u}) = 0} \quad \left(\partial_t \rho + \frac{\partial}{\partial x_k}(\rho u_k) = 0 \right)$$

$$\frac{D}{Dt}\rho + \rho \nabla \cdot u = 0 \quad \left(\partial_t \rho + u_k (\partial_k \rho) + \rho \partial_k u_k = 0 \right)$$

ρ constant along particle paths
$\Leftrightarrow \nabla \cdot u = 0$

Reynold's Transport Theorem

$$\frac{d}{dt}\int_V \alpha \, dV = \int_V \alpha_t + \nabla \cdot (\alpha \underset{\sim}{u}) \, dV$$

$$= \int_V \alpha_t + \frac{\partial}{\partial x_k}(\alpha u_k) \, dV$$

$$= \int_V \alpha_t + u_k \frac{\partial \alpha}{\partial x_k} + \alpha \frac{\partial u_k}{\partial x_k} \, dV$$

$$= \int_V \alpha_t + u \cdot \nabla \alpha + \alpha \nabla \cdot u \, dV$$

$$= \int_V \frac{d\alpha}{dt} + \alpha \nabla \cdot u \, dV$$

Conservation of Momentum

$$\frac{d}{dt}\int_V \rho u_i \, dV = \int_{\partial V} \sigma_{ji} n_j \, dA + \int_V \rho f_i \, dV$$

$$\int_V \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_k}(\rho u_i u_k) \, dV = \int_V \frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i \, dV$$

$$\rho \partial_t u_i + \underbrace{u_i \partial_t \rho + u_i \partial_k (\rho u_k)}_{0} + \rho u_k \partial_k u_i = \partial_j \sigma_{ji} + \rho f_i$$

$$\boxed{\rho \frac{\partial}{\partial t} u_i + \rho u_k \frac{\partial u_i}{\partial x_k} = \frac{\partial}{\partial x_j} \sigma_{ji} + \rho f_i}$$

$$\rho \, d_t \underset{\sim}{u} = \underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma} + \rho \underset{\sim}{f}$$

$$d_t \underset{\sim}{u} = \frac{1}{\rho} \underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma} + \underset{\sim}{f}$$

Conservation of Energy

Let e = internal energy per unit mass
q = conductive heat flux leaving control volume
P = pressure $= n \cdot \sigma$ $(P_j = n_i \sigma_{ij})$

$$\frac{D}{Dt}\int_V (\rho e + \tfrac{1}{2}\rho u \cdot u) \, dV = \int_S u \cdot P \, dS + \int_V u \cdot \rho f \, dV - \int_S q \cdot n \, dS$$

Using Gauss' Thm, conservation of mass, and conservation of momentum,
This simplifies to:

$$\boxed{\rho \frac{\partial e}{\partial t} + \rho u_k \frac{\partial e}{\partial x_k} = \sigma_{ij} \frac{\partial u_j}{\partial x_i} - \frac{\partial q_j}{\partial x_j}}$$

Derivation of Reynold's Transport Theorem

Assume that V is a control volume moving with a fluid that is moving with a velocity field $\mathbf{u}(\mathbf{x},t)$.

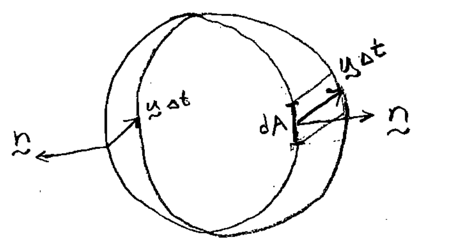
Let $\alpha(\mathbf{x},t)$ be a (scalar) quantity.

Let $I \equiv d_t \int_V \alpha \, dV$

$$I = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{V(t+\Delta t)} \alpha(t+\Delta t)\, dV - \int_{V(t)} \alpha(t)\, dV\right]$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\int_{V(t+\Delta t)} \alpha(t+\Delta t)\, dV - \int_{V(t+\Delta t)} \alpha(t)\, dV + \int_{V(t+\Delta t)} \alpha(t)\, dV - \int_{V(t)} \alpha(t)\, dV\right]$$

$$= \int_V \lim_{\Delta t \to 0} \frac{\alpha(t+\Delta t)-\alpha(t)}{\Delta t}\, dV + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\int_{V(t+\Delta t)-V(t)} \alpha(t)\, dV$$



$$= \int_V \alpha_t \, dV + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\int_{\partial V} \alpha (\mathbf{u}\Delta t)\cdot \mathbf{n}\, dA$$

$$= \int_V \alpha_t \, dV + \int_{\partial V} \alpha\, \mathbf{u}\cdot\mathbf{n}\, dA \quad \text{(intuitively evident)}$$

$$= \int_V \alpha_t + \nabla\cdot(\alpha \mathbf{u})\, dV$$

Shortcut to Reynold's transport

$$d_t \int_V \alpha\, dV = \int_V (d_t \alpha)\, dV + \alpha (d_t dV)$$

$$= \int_V \alpha_t + \mathbf{u}\cdot\nabla\alpha + \alpha \nabla\cdot\mathbf{u}\, dV$$

$$= \int_V \alpha_t + \nabla\cdot(\alpha \mathbf{u})\, dV$$

Derivation of Advection-Diffusion

Using convected control volume $V(t)$:

$$D_t \int_V \varphi = \int_{\partial V} K \hat{n}\cdot\nabla\varphi + \int_V P$$

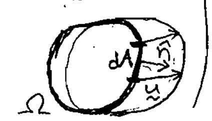
So $\int_V \varphi_t + \nabla\cdot(\varphi \mathbf{u}) = \int_V \nabla\cdot(K\nabla\varphi) + \int_V P$

So $\int_V \varphi_t + \nabla\cdot(\varphi \mathbf{u} - K\nabla\varphi) = \int_V P$

Using stationary control volume Ω:

$$\frac{d}{dt}\int_\Omega \varphi = \int_{\partial\Omega} -\varphi\, \hat{n}\cdot\mathbf{u}\, dA + \int_{\partial\Omega} \hat{n}\cdot K\nabla\varphi\, dA + \int_\Omega P$$

So $\int_\Omega \varphi_t + \nabla\cdot(\varphi\mathbf{u} - K\nabla\varphi) = \int_\Omega P$



General flux law:

$$\frac{d}{dt}\int_\Omega \varphi = \int_{\partial\Omega} -\hat{n}\cdot q + \int_\Omega P$$

So $\int_\Omega \varphi_t + \nabla\cdot q = \int_\Omega P$

Note: no concept of convection or diffusion assumed here. Just book-keeping. Choice of q determines P.

Hence Advection-Diffusion constitutive law

Set $q \equiv \varphi\mathbf{u} - K\nabla\varphi$.

Constitutive Equations

· Heat Flux $\underset{\sim}{q} = -k\underset{\sim}{\nabla} T$ k = thermal conductivity

· Stress-Strain

$$\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij}$$

$$\tau_{ij} = K_{ijmn} D_{mn}$$

$$= \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + 2\mu D_{ij}$$

$$\underset{\approx}{\tau} = \lambda \underset{\approx}{I}\, \underset{\sim}{\nabla}\cdot \underset{\sim}{u} + 2\mu \underset{\approx}{D}$$

(monatomic gases: $\lambda = -\frac{2}{3}\mu$)

So $\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k}$

Navier-Stokes (mom cons. + stress-strain)

$$\frac{\partial \sigma_{ij}}{\partial x_i} = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\lambda\frac{\partial u_k}{\partial x_k}\right) + \frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]$$

Assume $\boxed{\nabla\cdot u = 0}$ and μ constant.

So $\frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] = \mu\frac{\partial^2 u_j}{\partial x_i \partial x_i}$

μ is called the [dynamic] viscosity.

So $\frac{\partial \sigma_{ij}}{\partial x_i} = -\frac{\partial p}{\partial x_j} + \mu\frac{\partial^2 u_j}{\partial x_i\partial x_i}$

$$\boxed{d_t u = \frac{1}{\rho}\left[-\nabla p + \mu\nabla^2 u\right] + f}$$

If viscosity is zero, get Euler equations for inviscid fluid:

$$\boxed{d_t u = \frac{1}{\rho}(-\nabla p) + f}$$

$\nu \equiv \frac{\mu}{\rho}$ is called the kinematic viscosity.
$[\nu] = L^2T^{-1}$
$[\mu] = ML^{-1}T^{-1}$

Intuitive Derivation of Navier-Stokes

Convected test volume $(v \equiv u)$

$$\frac{d}{dt}\int_{\Omega(t)} \rho v_i = \int_{\partial\Omega} \text{force}$$

$$\text{LHS} = \int_\Omega (\rho v_i)_t + v\cdot\nabla(\rho v_i)$$

$$= \int_\Omega \rho(v_i)_t + \overset{0}{\underline{v_i\rho_t + v_i(v\cdot\nabla\rho)}} + \rho v\cdot\nabla v_i$$

$$\text{RHS} = \int_{\partial\Omega} -n_i p + \int_{\partial\Omega}\mu\,\hat n\cdot\nabla v_i + \int_\Omega f\rho$$

$$= \int_\Omega -\nabla p + \nabla\cdot(\mu\nabla v_i) + f\rho$$

Stationary test volume

(same RHS).

$$\text{LHS} = \frac{d}{dt}\int_\Omega \rho v_i + \int_{\partial\Omega}\hat n\cdot(\rho v_i v)$$

$$= \int_\Omega (\rho v_i)_t + \underbrace{\nabla\cdot(\rho v_i v)}_{\rho v_i \underbrace{\nabla\cdot v}_{=0} + v\cdot\nabla(\rho v_i)}$$

= same LHS as above.

$$\boxed{\rho(u + u\cdot\nabla u) = -\nabla p + \mu\nabla^2 u + f\rho}$$

Kinetics

velocity gradient tensor $= \frac{\partial u_i}{\partial x_j} = \partial_j u_i$

$$= \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}$$

$$= D_{ij} + \frac{1}{2}\Omega_{ij}$$

= strain rate tensor + $\underbrace{\tfrac{1}{2}\text{ vorticity tensor}}_{\text{rotation tensor } (R_{ij})}$

flow near a point

$$u(x+h) = u(x) + \nabla u(x)\cdot h + O(h^2)$$

$$\frac{dh}{dt} = \frac{d(x+h)}{dt} - \frac{dx}{dt}$$

$$= \nabla u(x)\cdot h + O(h^2)$$

$$\frac{dh}{dt} = \nabla u(x_0)\cdot h_0 + O(t) + O(h_0^2)$$

$$h = h_0 + \nabla u(x_0)\cdot h_0 t + O(t^2) + O(h_0^2 t)$$

$$h = [I + \nabla u(x_0)t]\cdot h_0 + O(t^2, th^2)$$

$$\underset{\sim}{h} = (\underset{\approx}{I} + \underset{\approx}{D}t)\cdot(\underset{\approx}{I} + \underset{\approx}{R}t)\cdot\underset{\sim}{h}_0 + O(t^2, th^2)$$

This means that we can follow what happens to the displacement vector h by successively solving the following two ODEs:

(1) $\frac{dh}{dt} = R\cdot h$, $h(t=0) = h_0$, $h(t=t) = h_1$

(2) $\frac{dh}{dt} = D\cdot h$, $h(t=0) = h_1$, $h(t=t) = h_{result}$.

This is an example of the technique of operator splitting.

2,1

Since D is symmetric, it has a set of orthogonal eigenvectors. D expands/contracts in these directions.

The matrix $R = \frac{1}{2}\Omega$ is antisymmetric. Write:

$$\Omega = \begin{bmatrix} 0 & \partial_2 u_1 - \partial_1 u_2 & \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 & 0 & \partial_3 u_2 - \partial_2 u_3 \\ \partial_1 u_3 - \partial_3 u_1 & \partial_2 u_3 - \partial_3 u_2 & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & -\omega_3 & \omega \\ \omega_3 & 0 & -\omega \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Observe:

$$\Omega\cdot h = \omega\wedge h = e_i\varepsilon_{ijk}\omega_j h_k$$

$$R\cdot h = \left(\tfrac{1}{2}\Omega\right)\cdot h = \left(\tfrac{1}{2}\omega\right)\wedge h.$$

ω is the vorticity, also known as the curl

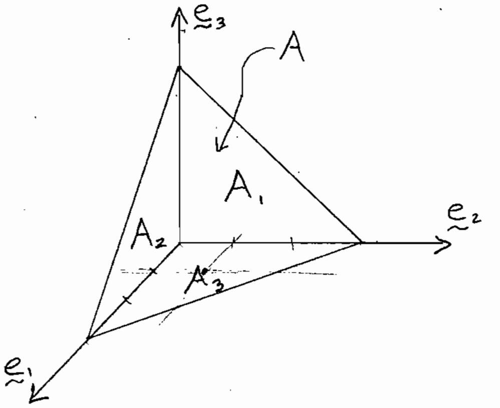
$$\omega = \nabla\wedge u = e_i\varepsilon_{ijk}\partial_j u_k = \begin{pmatrix}\partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1\end{pmatrix}$$

$\frac{1}{2}\omega$ is the rate of rotation.

So $R\cdot h = (\frac{1}{2}\Omega)\cdot h = (\frac{1}{2}\omega)\wedge h = (\frac{1}{2}\nabla\wedge u)\wedge h$

So $\boxed{\frac{dh}{dt} = D\cdot h + \left(\tfrac{1}{2}\nabla\wedge u\right)\wedge h \quad \text{where } D = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ for small } h}$

Derivation of stress tensor



Let $\tilde{p}_{-i}$ = pressure exerted by outside medium on the face of the tetrahedron orthogonal to $\tilde{e}_i$

Let $\tilde{p}_i = -\tilde{p}_{-i}$ = pressure exerted by tetrahedron on outside medium

Let $\hat{n}$ = unit normal to slanted surface pointing <u>outward</u> from tetrahedron.

Let $\tilde{p}_{\hat{n}}$ = pressure exerted by the rest of the medium on slanted surface A.

Observe that:

- $A_i = n_i A$
- Forces are:

  $\tilde{f}_i = \tilde{p}_i A_i$

  $\tilde{f}_{\hat{n}} = \tilde{p}_{\hat{n}} A$
- The sum of the forces on the tetrahedron equals its mass times its acceleration. The mass is proportional to the volume, and the forces are proportional to the surface area. As the tetrahedron shrinks to a point, the ratio of volume to surface area shrinks to zero. Thus, for an infinitesimal tetrahedron, the sum of the surface forces is 0:

  $$\tilde{f}_{\hat{n}} + \sum_i \tilde{f}_{-i} = 0$$

  i.e. $\tilde{f}_{\hat{n}} = \sum_i \tilde{f}_i$

  So $\tilde{p}_{\hat{n}} A = \sum_i \tilde{p}_i A_i = \sum_i \tilde{p}_i n_i A$

  So $\tilde{p}_{\hat{n}} = \sum_i n_i \tilde{p}_i$

Expand each vector in terms of its components. So write:

$$\tilde{p}_{\hat{n}} = \sum_k p_{\hat{n}k} \tilde{e}_k \quad \text{and} \quad \tilde{p}_i = \sum_k p_{ik} \tilde{e}_k$$

So $p_{\hat{n}k} = \sum_i n_i p_{ik}$, i.e. $\boxed{\tilde{p}_{\hat{n}} = \tilde{\hat{n}} \cdot \tilde{\tilde{P}}}$

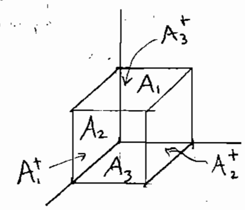
Note p_{ik} = k^{th} component of pressure exerted by the tetrahedron on the outside medium on the face $\perp$ to $\tilde{e}_i$

i.e. exerted by the positive side of an infinitesimal surface element $\perp$ to $\tilde{e}_i$ on its negative side.

Symmetry of stress tensor

- Observe that for an infinitesimal tetrahedron, the sum of the torques around any axis must be 0.
- Easier way: for an infinitesimal cube, the sum of the torques around any axis must be zero.

Picture:



For the purpose of calculating torque, a uniform force acting on a flat surface may be regarded as acting on its center of gravity.

WLOG choose the axis (of symmetry) passing through the centers of A_3 and A_3^+.

No forces contribute to torque except those coming from p_{12} and p_{21}.

Let A = area of each face.

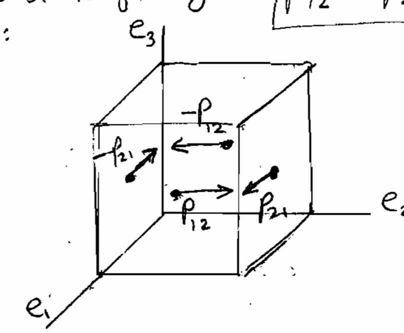
Let $\tilde{f}_i$ = force on $A_i = -\tilde{p}_i A$. So $f_{ik} = -p_{ik} A$

Let $\tilde{f}_i^+$ = force on $A_i = \tilde{p}_i A$. So $f_{ik}^+ = p_{ik} A$

So $f_{ik}^+ = -f_{ik}$, and $p_{ik}^+ = -p_{ik}$.

Balance of torques gives $\boxed{p_{12} = p_{21}}$

Picture:



Conservation Laws

Let u = conserved quantity. $(u(\underline{x},t))$

Let $\underline{f}$ = flux rate out $(f(u))$

Let p = production rate $(p(\underline{x},t))$

General conservation law.

(C) • **Integral form:**

Let Ω be a control volume in space.

(C1) $$\boxed{\frac{d}{dt}\int_\Omega u + \int_{\partial\Omega} \hat{n}\cdot f = \int_\Omega p}$$

scalar case:

$$\frac{d}{dt}\int_{x_1}^{x_2} u + [f]_{x_1}^{x_2} = \int_{x_1}^{x_2} p$$

Deriving differential form:

* Assume u and f are differentiable (as functions of $\underline{x}$ and t.)

Then $\int_\Omega u_t + \int_\Omega \nabla\cdot f = \int_\Omega p$

Assuming derivatives are continuous, since Ω is arbitrary, get:

(D) • **Differential form:**

$$\boxed{u_t + \nabla\cdot f = p}$$

scalar case:

$$u_t + f_x = p$$

Deriving variational formulation:

Let v be a test function with compact support. (continuous, sufficiently differentiable.)

$$\int_0^\infty dt \int_{\mathbb{R}^n} (u_t + \nabla\cdot f - p)\, v\, d\underline{x} = 0$$

$$\left(\int_{\mathbb{R}^n} d\underline{x} \int_0^\infty dt\, v u_t\right) + \int_0^\infty dt \int_{\mathbb{R}^n} v\, \nabla\cdot f\, d\underline{x} - \int_0^\infty dt \int_{\mathbb{R}^n} d\underline{x}\; pv = 0$$

$$-\int_{\mathbb{R}^n} d\underline{x}\,(vu)\big|_{t=0} - \int_{\mathbb{R}^n} d\underline{x} \int_0^\infty dt\; v_t u - \int \nabla v\cdot f - \int_0^\infty dt \int_{\mathbb{R}^n} d\underline{x}\; pv = 0$$

(V) • **Variational formulation**

$$\boxed{0 = \int_0^\infty \int_{\mathbb{R}^n} [v_t u + (\nabla v)\cdot f + vp]\, d\underline{x}\, dt + \int_{\mathbb{R}^n} (vu)_{t=0}\, d\underline{x}}$$

scalar case:

$$0 = \int_0^\infty \int_{-\infty}^\infty [v_t u + v_x f + vp]\, dx\, dt + \int_{-\infty}^\infty (vu)_{t=0}\, dx$$

Equivalence of integral form and variational form

(V) ⇒ (C)

Choose Ω.

Define $W_1(\underline{x}) = \begin{cases} 1 & \text{inside } \Omega \\ 0 & \text{outside } \Omega \text{ at a distance greater than } \varepsilon. \\ 1-\frac{d}{\varepsilon} & \text{at a distance } d \text{ from } \Omega,\ 0\le d\le\varepsilon \end{cases}$

Define $W_2(t) = \begin{cases} 1 & \text{in } [t_1,t_2]. \text{ Assume } t_1>0 \\ 0 & \text{outside } [t_1-\varepsilon,\ t_2+\varepsilon] \\ 1-\frac{d}{\varepsilon} & \text{at } t_1-d \text{ and } t_2+d \end{cases}$

Let $v(\underline{x},t) = W_1(\underline{x})\, W_2(t)$.

So $v_t = W_1(\underline{x})\, W_2'(t)$

$\nabla v = [\nabla W_1(\underline{x})]\, W_2(t)$

Note that W_2' and ∇W_1 are zero except at the boundary, where they look like spike functions:

$$W_1'\big|_{\partial[t_1,t_2]} = -\frac{1}{\varepsilon}\text{"}\hat{n}\text{"}$$

$$W_2'\big|_{\partial\Omega} = -\frac{1}{\varepsilon}\hat{n}.$$

So the variational formulation becomes:

$$0 = -\int_\Omega d\underline{x}\,[u]_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \int_{\partial\Omega} \hat{n}\cdot f + \int_{t_1}^{t_2}\int_\Omega p$$

(C2) $$\boxed{\int_\Omega [u]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\partial\Omega} \hat{n}\cdot f = \int_{t_1}^{t_2}\int_\Omega p}$$

(Another form of the integral form of a general conservation law.)

Scalar form:

$$\int_{x_1}^{x_2} u(x,t_2)\,dx - \int_{x_1}^{x_2} u(x,t_1)\,dx + \int_{t_1}^{t_2} f(x_2,t) - f(x_1,t)\,dt$$
$$= \int_{t_1}^{t_2}\int_{x_1}^{x_2} p(x,t)\,dx\,dt$$

Clearly (C1) ⇒ (C2). For (C2) ⇒ (C1), differentiate with respect to t_2. (Can differentiate $\int_\Omega [u]_{t_1}^{t}\,d\underline{x}$ because it equals something differentiable.)

(C2) ⇒ (V)

(C2) tells us that (V) is true for plateau functions like the one defined above.

(Indeed (C2) tells us that (V) is true for v with support boundary not including $t=0$. But then (V) must also be true for v with support boundary including $t=0$.)

Since every test function v with compact support can be approximated by a sum of plateau test functions, linearity lets us conclude that (V) is true for general test functions.

Convection of vorticity and circulation for inviscid flow when:

- the pressure depends on density alone, and
- any body forces are conservative

(Kelvin's circulation theorem.)

Thm Let $\underline{C}(t) = \underline{C}(s;t)$ denote a convected closed circuit:

① $\underline{C}(0,t) = \underline{C}(1,t)$

② $\forall s$, $\underline{C}(s;t)$ is convected with t.

i.e. $\partial_t C = \underline{u}(\underline{C}(s,t))$.

(Write $x(s,t) = \underline{C}(s,t)$.
$\underline{u}(s,t) = u(x(s,t))$)

Then the circulation

$$\Gamma = \int_{C(t)} \underline{u} \cdot dx$$

is independent of time.

Lemma (General). Let d_t = material derivative.

$$\boxed{d_t \int_{C(t)} u \cdot dx = \int_{C(t)} (d_t u) \cdot dx}$$

Pf (Acheson)

$$LHS = d_t \int_0^1 u(x(s;t)) \cdot \frac{\partial x}{\partial s} \cdot ds$$

$$= \int_0^1 \partial_t (\underline{u} \cdot \partial_s \underline{x})\, ds$$

$$= \int_0^1 \underbrace{(\partial_t \underline{u})}_{d_t u} \cdot \underbrace{(\partial_s \underline{x})}_{d\underline{x}} ds + \underbrace{\int_0^1 \underline{u} \cdot \partial_s \underbrace{\partial_t \underline{x}}_{u}\, ds}_{\underbrace{\partial_s(\frac{1}{2} u \cdot u)}_{0}}$$

$$= \int_{C(t)} (d_t \underline{u}) \cdot d\underline{x}$$

Pf (Currie)

$$d_t \oint \underline{u} \cdot d\underline{x} = \oint d_t (\underline{u} \cdot d\underline{x})$$

$$= \oint (d_t \underline{u}) \cdot d\underline{x} + \underbrace{\oint \underline{u} \cdot d_t d\underline{x}}_{\text{Call } Z\text{, want} = 0.}$$

$$Z = \oint \underline{u} \cdot d(d_t \underline{x})$$

$$= \oint u \cdot du$$

$$= \oint d(\tfrac{1}{2} u \cdot u)$$

$$= 0.$$

(Note: $df \equiv \nabla f \cdot dx$)

Pf of thm

Recall Euler's equation for conservative body force:

$$d_t u = -\frac{\nabla p}{\rho} - \nabla X$$

(1) Assume $p = p(\rho)$. (True for any compressible fluid)

Finding P s.t. $\nabla P = \frac{\nabla p}{\rho}$:

Seek $P = P(\rho)$

Need $\nabla P = \frac{\nabla p}{\rho}$

i.e. $P'(\rho) \nabla \rho = \frac{p'(\rho)}{\rho} \nabla \rho$

i.e. $P'(\rho) = \frac{p'(\rho)}{\rho}$

i.e. $\boxed{P(\rho) = \int \frac{p'(\rho)}{\rho} d\rho}$

Check:

$$\nabla P = P'(\rho) \nabla \rho$$
$$= \frac{p'(\rho)}{\rho} \nabla \rho \checkmark$$

(2) Assume $\rho = \rho(p)$ (works even when ρ is constant.)

Seek $R = R(p)$ s.t. $\nabla R = \frac{\nabla p}{\rho}$

Need $R'(p) \nabla p = \frac{\nabla p}{\rho}$

Need $R'(p) = \frac{1}{\rho}$

Need $\boxed{R(p) = \int \frac{dp}{\rho(p)}}$ (Note $R = P$.)

Check: $\nabla R = R'(p) \nabla p = \frac{\nabla p}{\rho} \checkmark$

i.e. $\frac{\nabla p}{\rho} = \nabla \int \frac{dp}{\rho}$

So can write:

$$d_t u = -\nabla\left(\int \frac{dp}{\rho} + X\right)$$

So $d_t \Gamma = d_t \int_{C(t)} \underline{u} \cdot d\underline{x}$

$$= \int_{C(t)} (d_t u) \cdot dx$$

$$= \int_{C(t)} -\nabla\left(\int \frac{dp}{\rho} + X\right) \cdot dx$$

$$= 0.$$

2D irrotational incompressible flow

① Does steady, irrotational flow automatically satisfy the Euler equations for incompressible, inviscid flow?

Yes if the flow is in the presence of a conservative body force.

② Flow that is initially irrotational remains irrotational under the following conditions:

- inviscid
- constant density (or $p = p(\rho)$)
- body forces are conservative

Euler

$$(E)\begin{cases} u_t + u\cdot\nabla u = -\frac{1}{\rho}\nabla p + \nabla\chi \\ \nabla\cdot u = 0 \end{cases} \forall x\in\Omega$$
$$u\cdot\hat{n} = B(x) \quad \forall x\in\partial\Omega$$

Irrotational:

$$(I)\begin{cases} \nabla\wedge u = 0 \\ \nabla\cdot u = 0 \end{cases} \forall x\in\Omega$$
$$u\cdot\hat{n} = B(x) \quad \forall x\in\partial\Omega$$

① Irrotational $\Rightarrow$ Euler.

Assume (I). Need (E).

Need $\exists p$ s.t.

$-(\nabla p)/\rho = [u_t + u\cdot\nabla u + \nabla\chi]$

[Assume that pressure is a function of density and density is a function of pressure.

Let $r(p) = \frac{-1}{\rho(p)}$

Write $R'(p) = r(p)$ (some R).

Note $-\frac{\nabla p}{\rho} = R'(p)\nabla p = \nabla(R(p))$

If we can find an R then we have an r and thus a p.]

(Easier case: ρ is constant.)

So we need to show that the RHS is the gradient of something.

So we need that the curl of the RHS is zero.

$$\nabla\wedge RHS = \underbrace{(\nabla\wedge u)_t}_{0} + \nabla\wedge(u\cdot\nabla u) + \underbrace{\nabla\wedge\nabla\chi}_{0}$$

Need $\nabla\wedge(u\cdot\nabla u) = 0$

But $u\cdot\nabla u = \underbrace{(\nabla\wedge u)}_{0}\wedge u + \nabla(\frac{1}{2}u\cdot u)$

So need $\nabla\wedge\nabla(\frac{1}{2}u\cdot u) = 0$. Clear. Done

② Euler + initially irrotational $\Rightarrow$ remains irrotational (Acheson 161)

Thm (Cauchy-Lagrange)

Let an inviscid fluid move in the presence of a conservative body force.
Let density depend on pressure alone.
Then if a portion of fluid is in irrotational motion it will remain in irrotational motion.

Pf

Invoke Kelvin's circulation theorem around some small region where the vorticity is supposedly nonzero.
By Stoke's theorem, the circulation will be nonzero, violating Kelvin:

$$\int_{\partial S} u\cdot dx = \int_S (\nabla\wedge u)\cdot\hat{n}\,dS$$