Notes on Curves (Chapter 13, Calculus of Vector-Valued Functions)

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Chapter 13 studies the properties of vector-valued functions, i.e. functions from the real line $\mathbb{R}$ to $\mathbb{R}^n$. We will use $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ to stand for such a vector-valued function into 3-space $\mathbb{R}^3$.

1 Calculus of vector-valued functions.

The calculus of vector-valued functions is very simple. The derivative is just the derivative of the components:

$$\begin{aligned}\mathbf{r}'(t) &= \lim_{dt\to 0} \frac{\mathbf{r}(t+dt) - \mathbf{r}(t)}{dt} \\ &= \langle x'(t), y'(t), z'(t)\rangle.\end{aligned}$$

The properties of derivatives are just like for single-variable calculus. For example, we have the product rules:

$$\begin{aligned}(s\mathbf{r})' &= s'\mathbf{r} + s\mathbf{r}', \\ (\mathbf{r}_1 \cdot \mathbf{r}_2)' &= \mathbf{r}_1' \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_2', \\ (\mathbf{r}_1 \times \mathbf{r}_2)' &= \mathbf{r}_1' \times \mathbf{r}_2 + \mathbf{r}_1 \times \mathbf{r}_2'.\end{aligned}$$

2 Parametrized curves

A curve C in space is specified by a vector-valued function $\mathbf{r}(t)$ from the real line $\mathbb{R}$ to space $\mathbb{R}^3$. The function $\mathbf{r}(t)$ is called a **parametrization** of C. We think of t as time. As the parameter t advances, $\mathbf{r}(t)$ traces out the curve in space and determines its orientation. (The *orientation of a curve* is the direction in which it is traversed. A given curve has one of two possible orientations.)

There are many ways to parametrize a given curve. Let $\tilde{t}(\tau)$ be a smooth one-to-one function. Then $\tilde{\mathbf{r}}(\tau) := \mathbf{r}(\tilde{t}(\tau))$ is a new parametrization, called a **reparametrization** of the same curve C, in terms of a new "time" τ. The chain rule works componentwise, and it tells us (in Newton's "function" notation) that

$$\tilde{\mathbf{r}}'(\tau) = \mathbf{r}'(\tilde{t}(\tau))\tilde{t}'(\tau). \tag{1}$$

3 Quantity notation.

When we are doing geometry we will often think in terms of physical *quantities* (the way physicists like to think) rather than in terms of functions (the way mathematicians like to think). In quantity notation, to indicate a function we write something like $\mathbf{r}(t)$, i.e., the quantity named $\mathbf{r}$ as a function of the quantity named t. In this notation, $\mathbf{r}(\tau)$ and $\mathbf{r}(t)$ are two different functions. To map the function notation above to the quantity notation here, we have

$$\begin{aligned}\tilde{t} &:= t(\tau), \\ \tilde{\mathbf{r}} &:= \mathbf{r}(\tau), \\ \mathbf{r} &:= \mathbf{r}(t).\end{aligned}$$

In this notation, we will not distinguish between $\tilde{\mathbf{r}}$ and $\mathbf{r}$ or between $\tilde{t}$ and t, since they represent the same physical quantity. In Leibnitz's "quantity" notation the chain rule (1) becomes:

$$\frac{d\mathbf{r}}{d\tau} = \frac{d\mathbf{r}}{dt}\frac{dt}{d\tau}.$$

Here $\frac{dt}{d\tau}$ stands for the derivative of the quantity named t as a function of the quantity named τ, and $\frac{d\mathbf{r}}{dt}$ stands for the derivative of the quantity named $\mathbf{r}$ as a function of the quantity named t. This notation does not make explicit the point where each quantity or derivative is evaluated. It is understood that each quantity is evaluated at the appropriate/natural/obvious point.

In quantity notation, when we want to specify where a function is evaluated we can use a vertical bar.

We write $\mathbf{r}(t)\big|_{t=t_1}$, or simply $\mathbf{r}|_{t=t_1}$ or $\mathbf{r}(t=t_1)$, to represent *the quantity named* $\mathbf{r}$ *as a function of the quantity named* t *evaluated at the point* t_1.

So to make the Leibnitz chain rule more explicit we can write:

$$\underbrace{\frac{d\mathbf{r}}{d\tau}\bigg|_{\tau=\tau_1}}_{\tilde{\mathbf{r}}'(\tau_1)} = \underbrace{\frac{d\mathbf{r}}{dt}\bigg|_{t=t(\tau_1)}}_{\mathbf{r}'(t(\tau_1))}\underbrace{\frac{dt}{d\tau}\bigg|_{\tau=\tau_1}}_{t'(\tau_1)}.$$

Quantity notation allows us to blur the distinction between the name of the input to a function and the point where we are evaluating the function. So $\mathbf{r}(t)$ may be used to mean $\mathbf{r}(t=t)$, i.e. the quantity $\mathbf{r}$ as a function of the quantity t evaluated at the point named t. Similarly, $\frac{d\mathbf{r}}{d\tau}$ is often understood to mean $\frac{d\mathbf{r}}{d\tau}\big|_{\tau=\tau}$, where we use τ to represent both the name of a quantity and a point where it is evaluated.

It is important for you to become proficient in translating between the more efficient quantity notation of the physicists and the more precise function notation of the mathematicians.

4 Curves

The goal of Chapter 13 is to study the *geometry* of curves. Geometry is the study of shape. *Shape* is a property of an object that is independent of the coordinates or parametrization one uses to describe it. For example, we will write down a formula for the radius of curvature of a curve. This formula should give the same answer no matter what parametrization or coordinate system we use to describe the curve.

Curved shapes are smooth. Calculus is basically the study of things that are smooth. Smooth means differentiable. So we use derivatives, i.e. rates of change, to study geometry.

We define the **velocity** $\mathbf{v}(t)$ to be the rate of change of position: $\mathbf{v}(t) := \frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$. We define the **acceleration** to be the rate of change of velocity: $\mathbf{a}(t) := \frac{d\mathbf{v}}{dt} = \mathbf{r}''(t) = \langle x''(t), y''(t), z''(t)\rangle$.

The starting point for understanding geometry is the notion of *arc length* and an *arc-length parametrization*. Let C be a curve with parametrization $\mathbf{r}(t)$. Pick a base time t_0. This defines a base point $\mathbf{r}_0 := \mathbf{r}(t_0)$. Let $s(t)$ denote the distance traced along the curve by $\mathbf{r}(t)$ from time t_0 to time t. The speed is $\frac{ds}{dt}$, i.e., the time-derivative of arc length. But speed is also $|\mathbf{r}'(t)|$, i.e., the magnitude of the velocity. So to find $s(t)$ we just solve the differential equation $\frac{ds}{dt} = |\mathbf{r}'(t)|$. We can solve for s by integrating:

$$s(t_1) - s(t_0) = \int_{s(t_0)}^{s(t_1)} ds = \int_{t_0}^{t_1} \frac{ds}{dt}dt, \text{ i.e.,}$$

$$\boxed{s(t) = \int_{t_0}^{t} |\mathbf{r}'(t)|dt}$$

Isaac Newton put a dot over a letter to denote the derivative with respect to time. So arc length is s, but speed is $\dot{s}$. Note the dot!

To summarize:

- A dot over a letter means the derivative with respect to t ("time"),
- $\mathbf{r}(t)$ = position as a function of time,
- $\mathbf{v} := \dot{\mathbf{r}}$ = velocity,
- $\mathbf{a} := \dot{\mathbf{v}} = \ddot{\mathbf{r}}$ = acceleration,
- s is distance along the curve (measured from some base point, and
- $\dot{s} := \frac{ds}{dt} = |\mathbf{v}|$ is the *speed.*

So here's the recipe to get an arc-length parametrization:

1. Find the speed $\dot{s} = \sqrt{\mathbf{r}'(t)\cdot\mathbf{r}'(t)}$.
2. Antidifferentiate to get $s(t)$, arc length as a function of time.
3. Invert to get $t(s)$, time as a function of arc length.
4. Plug $t(s)$ into $\mathbf{r}(t)$ to get the arc length parametrization $\mathbf{r}(t(s))$.

The essential property of an arc-length parametrization is that *the speed is 1*:

$$\left|\frac{d\mathbf{r}}{ds}\right| = \left|\frac{d\mathbf{r}}{dt}\right| \cdot \left|\frac{dt}{ds}\right| = 1,$$

since by definition $\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right|$. In other words:

$$\boxed{\frac{d\mathbf{r}}{ds}\cdot\frac{d\mathbf{r}}{ds} = 1}, \tag{2}$$

which is a useful result and a way to check your reparametrization.

For a given oriented curve C with base point $\mathbf{r}_0$, the arc-length parametrization will always be the same, regardless of the initial choice of parametrization. So we will define geometric quantities in terms of the arc-length parametrization, and this guarantees that our formulas are independent of parametrization.

5 First derivatives: velocity and unit tangent.

The velocity vector $\mathbf{v}(t)$ points in the direction of the curve. We call the unit vector in the direction of $\mathbf{v}$ the **unit tangent** vector $\hat{\mathbf{T}} := \widehat{\mathbf{v}}$, because it points in the direction of a tangent line to the curve. The velocity vector is its magnitude (the speed $\dot{s}$)) times its direction $\hat{\mathbf{T}}$:

$$\boxed{\mathbf{v} = \dot{s}\hat{\mathbf{T}}} \tag{3}$$

This is the geometric equation for first derivatives. A quick way to get this equation is to use the chain rule to differentiate position:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \underbrace{\frac{d\mathbf{r}}{ds}}_{\hat{\mathbf{T}}}\underbrace{\frac{ds}{dt}}_{\dot{s}} \tag{4}$$

If we parametrize by arc length, the speed is 1.

Exercise. Let $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 4t\rangle$. Find an arc-length parametrization of $\mathbf{r}(t)$. Verify that in the arc-length parametrization the speed is 1.

6 Second derivatives: acceleration and curvature.

Differentiating both sides of equation (3) with respect to time gives acceleration in terms of geometric quantities:

$$\begin{aligned}\mathbf{a} &= \frac{d\hat{\mathbf{T}}}{dt}\dot{s} + \hat{\mathbf{T}}\ddot{s} \text{ (by the product rule)} \\ &= \frac{d\hat{\mathbf{T}}}{ds}\frac{ds}{dt}\dot{s} + \hat{\mathbf{T}}\ddot{s} \text{ (by the chain rule)} \end{aligned} \tag{5}$$

The quantity $\frac{d\hat{\mathbf{T}}}{ds}$ represents the rate of change of direction with respect to arc length. It is defined in terms of the arc length, so it has a geometrical meaning. We call it the *curvature vector* $\vec{\kappa}$, because it points in the direction in which the curve is curving.

If we differentiate equation (2) (with respect to s) and use the product rule, we get:

$$2\underbrace{\frac{d\mathbf{r}}{ds}}_{\hat{\mathbf{T}}}\cdot\underbrace{\frac{d^2\mathbf{r}}{(ds)^2}}_{\vec{\kappa}} = 0,$$

which says that the curvature vector is always perpendicular to the unit tangent vector.

$\kappa := |\vec{\kappa}|$, the magnitude of the curvature vector, is called the **curvature**. It represents the rate of change of direction with respect to arc length. The reciprocal of the curvature is called the **radius of curvature**, $R = 1/\kappa$. The direction of the curvature vector, $\hat{\mathbf{N}} := \vec{\kappa}/|\vec{\kappa}|$, is called the unit normal to the curve. So we can rewrite equation (5) as:

$$\mathbf{a} = \underbrace{\vec{\kappa}\dot{s}^2}_{\mathbf{a}_N} + \underbrace{\hat{\mathbf{T}}\ddot{s}}_{\mathbf{a}_T},$$

where $\mathbf{a}_T$ and $\mathbf{a}_N$ denote the tangential and normal components of the acceleration.

In summary:

- $\hat{\mathbf{T}} := \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|}$ is the unit tanget vector,
- $\vec{\kappa} := \frac{d\hat{\mathbf{T}}}{ds}$ is the curvature vector,
- $\kappa = |\vec{\kappa}|$ is the curvature, and
- $\hat{\mathbf{N}} := \frac{\vec{\kappa}}{|\vec{\kappa}|}$ is the unit normal vector (i.e. the direction of the curvature vector),
- $\vec{\kappa} = \kappa\hat{\mathbf{N}}$, and
- $R := 1/\kappa$ is the radius of curvature.

7 Finding geometric quantities without reparametrizing

Reparametrizing is hard. With the exception of some very nice curves (e.g. helixes), most curves do not have an elementary arc-length parametrization. Fortunately, you can calculate all the geometric quantities (unit tanget, speed, tangential and normal components of acceleration, curvature, and unit normal) simply in terms of $\mathbf{v}$ and $\mathbf{a}$ without ever having to write down an arc-length parametrization explicitly. Here's how:

1. Compute the velocity $\mathbf{v} = \mathbf{r}'(t)$.
2. Compute the speed $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.
3. Compute the unit tangent $\hat{\mathbf{T}} = \frac{\mathbf{v}}{|\mathbf{v}|}$.
4. Compute the acceleration $\mathbf{a} = \mathbf{v}'(t)$.
5. Compute the scalar tangential acceleration $a_T = \mathbf{a} \cdot \hat{\mathbf{T}}$.
6. Compute the vector tangential acceleration $\mathbf{a}_T = a_T \hat{\mathbf{T}}$.
7. Compute the normal component of the acceleration $\mathbf{a}_N = \mathbf{a} - \mathbf{a}_T$.
8. Compute the curvature $|\vec{\kappa}| = \frac{|\mathbf{a}_N|}{|\mathbf{v}|^2} = \frac{\sqrt{|\mathbf{a}|^2 - a_T^2}}{|\mathbf{v}|^2}$.
9. Compute the radius of curvature $R = 1/|\vec{\kappa}|$.
10. Compute the unit normal $\hat{\mathbf{N}} = \frac{\mathbf{a}_N}{|\mathbf{a}_N|}$.

This method would work in n-dimensional space where n is any number. But in 2- and 3-dimensional space there are shortcuts.

In three dimensions the cross product provides a shortcut to find the magnitude of the curvature. Since $\mathbf{a}_T$ is parallel to $\hat{\mathbf{T}}$, $\mathbf{a}_T \times \hat{\mathbf{T}} = 0$, so:

$$|\mathbf{a}_N| = |\mathbf{a}_N \times \hat{\mathbf{T}}| = |(\mathbf{a}_N + \mathbf{a}_T) \times \hat{\mathbf{T}}|,$$

so

$$\boxed{|\vec{\kappa}| = \frac{|\mathbf{a} \times \mathbf{v}|}{|\mathbf{v}|^3}}.$$

In two dimensions we can simplify further. Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$. We regard two-dimensional space as a slice of three-dimensional space where the third component is zero. This is the natural *embedding* of two-dimensional space in three-dimensional space. So we write $\mathbf{r} = \langle x, y, 0 \rangle$. Then $\mathbf{v} = \langle \dot{x}, \dot{y}, 0 \rangle$ and $\mathbf{a} = \langle \ddot{x}, \ddot{y}, 0 \rangle$, so $\mathbf{v} \times \mathbf{a} = \langle 0, 0, \dot{x}\ddot{y} - \ddot{x}\dot{y} \rangle$, so

$$\boxed{|\vec{\kappa}| = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{\sqrt{\dot{x}^2 + \dot{y}^2}^3}}.$$

In three dimensions there is also a shortcut to find the unit normal $\hat{\mathbf{N}}$. The *unit binormal* $\widehat{\mathbf{B}}$ is defined so that $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, $\widehat{\mathbf{B}}$ is an ordered triple of three mutually orthogonal vectors with right-hand orientation. That is, $\widehat{\mathbf{B}} = \hat{\mathbf{T}} \times \widehat{\mathbf{B}}$. $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ span the same plane as $\mathbf{v}$ and $\mathbf{a}$. So $\mathbf{v} \times \mathbf{a}$ points in the same direction as $\widehat{\mathbf{B}}$. So $(\mathbf{v} \times \mathbf{a}) \times \mathbf{v}$ (which you can show equals $(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}$) points in the same direction as $\hat{\mathbf{N}}$.

8 Exercises

1. **Radius of curvature of a circle.** Since the radius of curvature already has a definition for circular paths, we need to show that our new definition of curvature agrees with the old definition.
   (a) Write a parametrization $\mathbf{r}(t)$ of a circle of radius R. (Hint: a parametrization of the unit circle is $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$.
   (b) Calculate the curvature κ.
   (c) Show that radius of curvature is indeed $1/R$.
2. **Arc-length parametrization of a helix.** A helix is a spiral. It looks like a slinky. It is produced by circular motion in a plane combined with linear motion in the perpendicular direction. For example, let $\mathbf{r}(t) = \langle R\sin(t), R\cos(t), at \rangle$.
   (a) Find the velocity and speed.
   (b) Find an arc-length parametrization of $\mathbf{r}$.
   (c) Find the unit tangent.
   (d) Find the acceleration.
   (e) Find the tangential and normal components of the acceleration.
   (f) Find the curvature and radius. Does you answer agree with the radius of a circle?
3. **Components of acceleration.** Let $\mathbf{r}(t) = \langle \ln(\cos(t)), t \rangle$. At time $t = \pi/3$ find:
   (a) the velocity,
   (b) the unit tangent,
   (c) the acceleration,
   (d) the components of the acceleration parallel and perpendicular to the unit tangent,
   (e) the curvature, and
   (f) the unit normal.