Course notes for vector field integration (§§16.7–8): Gauss's theorem and Stokes' theorem
by Alec Johnson, December 2009

This note generalizes Green's theorem (the fundamental theorem of calculus in the plane) to three dimensions.

1 Review

Recall that Green's theorem in the plane relates an integral around the boundary of a region to an integral of a derivative over the interior of the region. As a computational device its two ingredients were:

$$\oint N\,dy = \iint \frac{\partial N}{\partial x}\,dx\,dy,$$
$$\oint M\,dx = \iint -\frac{\partial M}{\partial y}\,dx\,dy.$$

Green's theorem had two interpretations or physical manifestations: a divergence theorem,

$$\oint_{\partial R} M\,dy - N\,dx = \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,dA, \text{ i.e.,}$$
$$\oint_{\partial R} \mathbf{F}\cdot\widehat{\mathbf{n}}\,ds = \iint_R \nabla\cdot\mathbf{F}\,dx\,dy,$$

and a circulation theorem,

$$\oint_{\partial R} M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA, \text{ i.e.,}$$
$$\oint_{\partial R} \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R \widehat{\mathbf{k}}\cdot\nabla\times\mathbf{F}.$$

2 Gauss's divergence theorem

In words the divergence theorem (in two dimensions) says that **the flux of a vector field out of the boundary of a region equals the integral of the divergence of the vector field over the interior of the region.** The divergence theorem in three dimensions says exactly the same thing! In symbols:

$$\boxed{\oiint_{\partial R} \mathbf{F}\cdot\widehat{\mathbf{n}}\,dA = \iiint_R \nabla\cdot\mathbf{F}\,dV},$$

where the region R is now a volume, the symbol $\oiint_{\partial R}$ signifies a boundary integral, and $\widehat{\mathbf{n}}$ is the *outward* unit normal vector.

Suppose $\mathbf{F} = (M, N, P)$ and $\widehat{\mathbf{n}} = (n_1, n_2, n_3)$. Then the divergence theorem is the sum of three identical-looking theorems:

$$\oiint_{\partial R} n_1 M\,dA = \iiint_R \partial_x M\,dV,$$
$$\oiint_{\partial R} n_2 N\,dA = \iiint_R \partial_y N\,dV,$$
$$\oiint_{\partial R} n_3 P\,dA = \iiint_R \partial_z P\,dV.$$

In other words, **Gauss's divergence theorem says that in an integral over a boundary you can replace components of $\widehat{\mathbf{n}} = (n_1, n_2, n_3)$ with the corresponding components of $\nabla = (\partial_x, \partial_y, \partial_z)$ if you change the boundary integral to an integral over the interior.** Which means that the divergence theorem can be used for much more than just divergences.

2.1 Proof of Gauss's theorem. To show that Gauss's theorem is true we just need to verify that

$$\oiint_{\partial R} n_1 M\,dA = \iiint_R \frac{\partial M}{\partial x}\,dV.$$

We will calculate the right hand side as an iterated integral and show that it is the same as the left hand side. So assume that R may be represented as the region between a lower surface $x_1(y,z)$ and an upper surface $x_2(y,z)$. (Why is there no loss of generality in this assumption? *Hint:* Any well-behaved region can be divided up into such regions; what can you say about outward fluxes along the common boundary between two such regions?) Let A be the projection of R onto the y-z plane (that is, A is the shadow of the volume R on the y-z plane when it is illuminated by rays traveling along the x axis). Then, making use of the fundamental theorem of calculus in one dimension,

$$\iiint_R \frac{\partial M}{\partial x}\,dV = \iint_A \int_{x_1(y,z)}^{x_2(y,z)} \frac{\partial M}{\partial x}\,dx\,dy\,dz$$
$$= \iint_A [M]_{x=x_1(y,z)}^{x_2(y,z)}\,dy\,dz.$$
$$= \iint_A M|_{x_2(y,z)}\,dy\,dz - \iint_A M|_{x_1(y,z)}\,dy\,dz.$$

We will get this same thing if we write the left hand side as an integral over the upper and lower surfaces. Here are two ways to see it:

(1) Analyze the area element geometrically. Each element $dx\,dy$ is the projection onto the y-z plane of a corresponding surface element $\widehat{\mathbf{n}}\,dA$. We need to show that on the upper surface $n_1\,dA = dy\,dz$ and that on the lower surface $n_1\,dA = -dy\,dz$. Since $\widehat{\mathbf{n}}$ is perpendicular to the surface element dA and $\widehat{\mathbf{i}}$ is perpendicular

to the surface element $dx\,dy$, the angle θ between the surface elements equals the angle between $\widehat{\mathbf{n}}$ and $\widehat{\mathbf{i}}$ (draw a picture). So $|n_1\,dA| = |\widehat{\mathbf{n}}\cdot\widehat{\mathbf{i}}\,dA| = |\cos\theta\,dA| = |dx\,dy|$. For the upper surface n_1 is positive, and for the lower surface n_1 is negative. Since dA and $dx\,dy$ are positive, $n_1\,dA = dy\,dz$ on the upper surface and $n_1\,dA = -dy\,dz$ on the lower surface, as needed.

(2) Use the natural parametrization and compute. The surface integral is the sum of integrals over the upper surface $x = x_2$ and the lower surface $x = x_1$. On the upper surface

$$\begin{aligned}
\iint_{x_2} n_1 M\,dA &= \iint_{x_2} \widehat{\mathbf{n}}\cdot(M,0,0)\,dA \\
&= \iint_A \begin{pmatrix} M \\ 0 \\ 0 \end{pmatrix} \cdot \frac{\partial}{\partial y}\begin{pmatrix} x_2 \\ y \\ z \end{pmatrix} \times \frac{\partial}{\partial z}\begin{pmatrix} x_2 \\ y \\ z \end{pmatrix} dy\,dz \\
&= \iint_A \begin{pmatrix} M \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_y x_2 \\ 1 \\ 0 \end{pmatrix} \times \frac{\partial}{\partial z}\begin{pmatrix} \partial_z x_2 \\ 0 \\ 1 \end{pmatrix} dy\,dz \\
&= \iint_A M\,dy\,dz, \qquad \text{as needed.}
\end{aligned}$$

Theoretical Exercise. Show that for the lower surface

$$\iint n_1 M\,dA = -\iint_A M\,dy\,dz.$$

(Hint: the parametrization must be properly oriented.)

2.2 Significance of Gauss's divergence theorem.

As you should recall from our study of Green's theorem, the divergence theorem reveals the physical meaning of the divergence. Considering a sufficiently small test volume (small enough so that $\nabla\cdot\mathbf{F}$ is approximately constant), it says that *the divergence is the outgoing flux per volume.*

The divergence theorem also allows us to show that the curl has a physical meaning. Recall that we defined the curl using a symbolic cross product: $\mathrm{curl}(\mathbf{F}) := \nabla\times\mathbf{F}$. This should have made you ask, "how do I know that the curl has any physical meaning"? In other words, "how do I know that if I calculate the curl in one system of coordinates I will get the same physical vector as if I calculate it in another set of coordinates?" Gauss's theorem allows us to show that the curl is *geometrically* defined (i.e. is independent of the choice of coordinate system). Gauss tells us that $\widehat{\mathbf{n}}$ can be replaced with ∇, so we can write:

$$\iiint_R \nabla\times\mathbf{F}\,dV = \oiint_{\partial R} \widehat{\mathbf{n}}\times\mathbf{F}\,dA.$$

Again, consider a test region R small enough so that $\nabla\times\mathbf{F}$ is approximately constant. I don't know what $\oiint_{\partial R}\widehat{\mathbf{n}}\times\mathbf{F}$ means physically, but I know it has a physical meaning, because everything in its definition is geometrically defined. Let's just call it the bliggelshig. Gauss tells us that the curl is the amount of bliggelshig per volume.

I remark that if you are given a formula in coordinates for a vector there is a more general way to check whether the vector has a physical meaning: compute it in two different coordinate systems and see if it represents the same physical vector. (If so, physicists say that the vector is "tensorial"—that is, it transforms properly under change of coordinates.) In a course in linear algebra or tensor calculus you would learn how to change coordinates from one system to another so that you can make such a check. With these tools it is straightforward to check that the cross product is tensorial.

3 Stokes' circulation theorem

Green's theorem states that the flux of the curl through a region of the x-y plane is the circulation around its boundary:

$$\oint_{\partial R} M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA, \text{ i.e.,}$$
$$\oint_{\partial R} \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R \widehat{\mathbf{k}}\cdot\nabla\times\mathbf{F}.$$

To generalize to other surfaces in three dimensional space, we first replace $\widehat{\mathbf{k}}$ (which is defined in terms of coordinates) with $\widehat{\mathbf{n}}$ (which is geometrically defined to be the normal vector to the plane):

$$\oint_{\partial R} \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R \widehat{\mathbf{n}}\cdot\nabla\times\mathbf{F}\,dA.$$

Everything in this formula is geometrically defined (including the curl), so Green's circulation theorem is actually a statement about any flat surface. Note that the direction of circulation and the orientation of the normal vector $\widehat{\mathbf{n}}$ must be consistently defined so that they satisfy the "right hand" rule: *if you make your graph with a right-handed coordinate system, the circulation should wrap in the direction of your fingers when your thumb points in the direction of* $\widehat{\mathbf{n}}$. (Recall that a coordinate system is right-handed if the fingers on your right hand curl from the x toward the y axis when your thumb points along $\widehat{\mathbf{k}}$).

The final step step in generalizing the circulation theorem to three dimensions is to realize that the surface does not need to be flat; it only needs to be smooth (or just piecewise smooth).

To see this, take a smooth surface S and chop it up into small pieces S_i that are approximately flat.

The circulation theorem is true on each flat piece:

$$\oint_{\partial S_i} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_{S_i} \widehat{\mathbf{n}} \cdot \nabla \times \mathbf{F}\, dA.$$

Now we sum this equation over all pieces i. The sum of the circulations of the pieces is the circulation around the boundary of the whole surface S, because the contribution of each shared edge to the circulation of its neighbors cancels in the sum (since the shared edge is traversed in opposite directions). So we have **Stokes' theorem**:

$$\boxed{\oint_{\partial S} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_{S} \widehat{\mathbf{n}} \cdot \nabla \times \mathbf{F}, dA},$$

which says that **the circulation of a vector field around the boundary of any "simple surface" (e.g. simple enough that you can parametrize it with a simply connected region) is the flux of the curl through the surface in the direction positively oriented with respect to the direction of circulation.**

Easy Theoretical Exercise. Generalize Stokes' theorem to a surface with a hole. Hint: chop up the region into pieces without holes.

Theoretical Exercise. Verify that Stokes' theorem is true by using a parametrization of an arbitrary surface.

Solution: Let $\mathbf{r}(u,v)$ be a parametrization mapping a region R in the plane to a surface S in space. Consider the special case $\mathbf{F} = M\widehat{\mathbf{i}}$. Using subscripts to denote partial derivatives, the flux integral is

$$\begin{aligned}
&\iint_S \widehat{\mathbf{n}}\, dA \cdot \nabla \times M\widehat{\mathbf{i}} \\
&= \iint_R (\mathbf{r}_u \times \mathbf{r}_v) \cdot \left(M_z\widehat{\mathbf{j}} - M_y\widehat{\mathbf{k}}\right) du\, dv \\
&= \iint_R \begin{pmatrix} y_u z_v - z_u y_v \\ z_u x_v - x_u z_v \\ x_u y_v - y_u x_v \end{pmatrix} \cdot \left(M_z\widehat{\mathbf{j}} - M_y\widehat{\mathbf{k}}\right) du\, dv \\
&= \iint_R -M_y(x_u y_v - y_u x_v) + M_z(z_u x_v - x_u z_v)\, du\, dv.
\end{aligned}$$

We can express the circulation integral in u and v coordinates and then apply Green's theorem to get the same thing:

$$\begin{aligned}
&\oint_{\partial S} M\widehat{\mathbf{i}} \cdot d\mathbf{r} \\
&= \oint_{\partial S} M\, dx = \oint_{\partial R} M(x_u\, du + x_v\, dv) \\
&= \iint_R -(Mx_u)_v + (Mx_v)_u\, du\, dv \\
&= \iint_R -M_v x_u + M_u x_v\, du\, dv \\
&= \iint_R -(M_x x_v + M_y y_v + M_z z_v) x_u\, du\, dv \\
&\quad + \iint_R (M_x x_u + M_y y_u + M_z z_v) x_v\, du\, dv \\
&= \iint_R -M_y(x_u y_v - y_u x_v) + M_z(z_u x_v - x_u z_v)\, du\, dv.
\end{aligned}$$

So Stokes' theorem holds for $\mathbf{F}_M := (M, 0, 0)$. A similar argument show that Stokes' theorem holds for $\mathbf{F}_N := (0, N, 0)$ and $\mathbf{F}_P := (0, 0, P)$. In fact, this follows just by rotating coordinates, because rotation of coordinates really is a $120°$ physical rotation around the line $x = y = z$ and the physical meaning of Stokes' theorem remains the same under this rotation of coordinates. Adding up the Stokes theorem for each of these vector fields shows that Stokes' theorem holds for $\mathbf{F} := \mathbf{F}_M + \mathbf{F}_N + \mathbf{F}_N = (M, N, P)$, as needed.

4 Application

Gauss's theorem and Stokes' theorem can be used to simplify surface integrals and work integrals. *To apply Gauss's theorem you must be working with a surface that is the boundary of a volume. To apply Stokes' theorem you must be working with a circulation integral or with the flux of a vector that is the curl of another vector field.*

If $\mathbf{F} = \nabla \times \mathbf{G}$, we say that $\mathbf{G}$ is a **vector potential** for $\mathbf{F}$. How can you test if $\mathbf{F}$ has a vector potential? Well, you can easily verify that the divergence of the curl of a vector field is zero. (For example, $\nabla \cdot \nabla \times M\widehat{\mathbf{i}} = \nabla \cdot (\partial_z M\widehat{\mathbf{j}} - \partial_y M\widehat{\mathbf{k}}) = \partial_y \partial_z M - \partial_z \partial_y M = 0$.) So if $\mathbf{G}$ has a vector potential, then its divergence must be zero. (This is in fact a sufficient condition to ensure that a vector potential exists.)

If a vector field $\mathbf{F}$ has a vector potential then Stokes' theorem says that the flux of $\mathbf{F}$ through any surface is completely determined by the location of the boundary. *So you are free to use a (nicer) surface to calculate the flux, as long as it has the same boundary!* In other words, *flux integrals of vector fields with a vector potential are "surface-independent", just as work integrals of vector fields* *with scalar potentials are "path-independent".* A vector field has a scalar potential if its curl is zero, and it has a vector potential if its divergence is zero.

Remark: The *Hodge decomposition theorem* says that *every vector field can be written as a sum of the curl of a vector potential and the gradient of a scalar potential.* Finding these potentials involves knowing how to solve a partial differential equation called "Poisson's equation".

Exercise. Let S be the hemispherical surface $x^2 + y^2 + z^2 = 1, \quad z \geq 0$, i.e. the upper half of the unit sphere. Find the upward flux of the curl of the vector field $\mathbf{F} = (-y^3, x^3, 0)$ through S.

Solution: $\nabla \times \mathbf{F} = 3(x^2 + y^2)\widehat{\mathbf{k}}$. To compute the flux directly, we need a parametrization. The most natural way to parametrize a sphere is to use spherical coordinates. Using subscripts to denote partial derivatives, $\mathbf{r}(\phi, \theta)$ and its derivatives are:

$$\begin{aligned} x &= \sin\phi\cos\theta, & x_\phi &= \cos\phi\cos\theta, & x_\theta &= -\sin\phi\sin\theta, \\ y &= \sin\phi\sin\theta, & y_\phi &= \cos\phi\sin\theta, & y_\theta &= \sin\phi\cos\theta, \\ z &= \cos\phi, & z_\phi &= -\sin\phi, & z_\theta &= 0. \end{aligned}$$

The domain R of $\mathbf{r}(\phi, \theta)$ is $0 \leq \phi \leq \pi/2, \ 0 \leq \theta \leq 2\pi$.

$$\iint_S dA\, \widehat{\mathbf{n}} \cdot \nabla \times \mathbf{F} = \iint_R \mathbf{r}_\phi \times \mathbf{r}_\theta \cdot \nabla \times \mathbf{F}\, d\phi\, d\theta$$

$$= \iint_R \begin{pmatrix} \sin^2\phi\cos\theta \\ \sin^2\phi\sin\theta \\ \cos\phi\sin\phi \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 3\sin^2\phi \end{pmatrix} d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} 3\cos\phi\sin^3\phi\, d\phi\, d\theta = \frac{3\pi}{2}.$$

Easier solution: Use Stokes' theorem. A parametrization of the boundary is $\mathbf{r}(t) = (\cos t, \sin t, 0), \ 0 \leq t \leq 2\pi$.

$$\iint_S dA\, \widehat{\mathbf{n}} \cdot \nabla \times \mathbf{F} = \oint_{\partial S} dr \ \cdot \mathbf{F} = \oint_{\partial S} -y^3\, dx + x^3\, dy$$

$$= \int_0^{2\pi} -y^3\, x'(t) + x^3\, y'(t)\, dt = \int_0^{2\pi} \sin^4 t + \cos^4 t\, dt = \frac{3}{2}\pi$$

(You can save some work if you note that the average value of a sinusoid is zero over any half cycle, and that the double angle identities imply that the average of $\sin^2$ or $\cos^2$ is one half over any half cycle.)

Even easier solution: Choose a better surface with the same boundary. Let D be the unit disc $x^2 + y^2 \leq 1$ in the x-y plane.

$$\iint_S dA\, \widehat{\mathbf{n}} \cdot \nabla \times \mathbf{F} = \oint_{\partial S} dr \ \cdot \mathbf{F} = \iint_D dA\, \widehat{\mathbf{k}} \cdot \nabla \times \mathbf{F}$$

$$= \iint_D dA\, 3(x^2 + y^2) = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 3r^2 r\, dr\, d\theta = \frac{3}{2}\pi.$$

If you need to find the flux of a vector field that does not have a vector potential through a surface which is not the boundary of a region, you can still often make use of Gauss's theorem to simplify the integral, but you have to be a little more clever:

Exercise. Let S be the upper half of the unit sphere. Find the upward flux of the vector field $\mathbf{G} = (0, y, z+1)$ through S.

Solution: Using the same parametrization as for the previous problem,

$$\iint_S dA\, \widehat{\mathbf{n}} \cdot \mathbf{G} = \iint_R \mathbf{r}_\phi \times \mathbf{r}_\theta \cdot \mathbf{G}\, d\phi\, d\theta$$

$$= \iint_R \begin{pmatrix} \sin^2\phi\cos\theta \\ \sin^2\phi\sin\theta \\ \cos\phi\sin\phi \end{pmatrix} \cdot \begin{pmatrix} 0, \\ \sin\phi\sin\theta \\ \cos\phi + 1 \end{pmatrix} d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \sin^3\phi\sin^2\theta + \cos^2\phi\sin\phi + \cos\phi\sin\phi\, d\phi\, d\theta.$$

Separating variables and using pythagorean identities, this becomes

$$\int_{\theta=0}^{2\pi} \sin^2\theta\, d\theta \int_{\phi=0}^{\pi/2} (1 - \cos^2\phi)\sin\phi$$

$$+2\pi \int_{\phi=0}^{\pi/2} (\cos^2\phi + \cos\phi)\sin\phi\, d\phi = \frac{7}{3}\pi$$

Easier solution: In general you can use the divergence theorem to transform a flux integral over a complicated surface into a flux integral over a simpler surface with the same boundary. For this problem, the boundary of the surface S is just a circle in the x-y plane. Another surface with this same boundary is just the unit disc D in the x-y plane. These two surfaces form a sandwich which is the boundary of the solid hemisphere H. Gauss's theorem tells us that the flux out of the solid hemisphere is the integral of the divergence of G over the interior. So to calculate the flux out of the top of the sandwich we use Gauss to calculate the flux out of the whole sandwich and then subtract off the flux out of the bottom surface (which is much easier to calculate). $\nabla \cdot \mathbf{G} = 2$, so $\iiint_H \nabla \cdot \mathbf{G} = \frac{4}{3}\pi$. The flux through the bottom surface is $\iint_D \widehat{\mathbf{k}} \cdot \mathbf{G} = \iint_D 1 = \pi$. So the flux through the top surface must be $\frac{4}{3}\pi + \pi = \frac{7}{3}\pi$.

Question. In my solutions to the exercises did my parametrization agree with the direction of my normal vector? How would you change the parametrized integral if you wanted to compute the downward flux?

Exercise. The last two exercises in the previous set of notes (on surface integrals) are trivial with Gauss's theorem. Redo them.