Course notes for multivariable integration (chapter 16)
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1 Overview

Chapter 16 is concerned with the integration of vector fields over curves and surfaces.

(We remark that curves and surfaces are examples of manifolds. *A manifold is* smooth*—that is, locally (i.e. on a small scale) a manifold looks flat. Curves locally look like straight lines; curves are also called* 1-dimensional manifolds. *Surfaces locally look like planes; surfaces are also called* 2-dimensional manifolds. *In a course in* differential manifolds *you would study how to do integration on higher-dimensional manifolds.)*

1.1 Types of multivariables integrals

Section	Type of integral	type of domain	type of integrand	form
16.1	**line** integral	curve	scalar field	$\int_C f\,ds$
16.2	**work** integral	curve	vector field	$\int_C \mathbf{F}\cdot d\mathbf{r}$
16.2	**circulation** integral	closed curve	vector field	$\oint_C \mathbf{F}\cdot d\mathbf{r}$
16.2	**flux** integral	(closed) curve	vector field	$\oint_C \mathbf{F}\cdot \hat{\mathbf{n}}\,ds$
16.5-6	**surface** integral	surface	scalar field	$\iint_S f\,dS$
16.5-6	**flux** integral	surface	vector field	$\iint_S \mathbf{F}\cdot \hat{\mathbf{n}}\,dS$

1.2 Forms of integrals for parametrized manifolds

To evaluate an integral over a curve or surface it is generally necessary to *parametrize* it.

1.2.1 Curve integrals:

Assume that C is an oriented curve that runs from $\mathbf{r}_a$ to $\mathbf{r}_b$, and let $\mathbf{r}(t), a \le t \le b$ parametrize curve C.

A line integral is a weighted length integral, a work integral is the work done by a vector field along an oriented path, and a flux integral over a closed loop is the net outward flow of the vector field:

$$\text{Line:}\quad \int_C f ds = \int_C f|d\mathbf{r}| = \int_a^b f\,|\mathbf{r}'(t)|\,dt$$

$$\text{Work:}\quad \int_C \mathbf{F}\cdot\hat{\mathbf{T}}ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_a^b \left(M\frac{dx}{dt}+N\frac{dy}{dt}\right)dt = \int_C M\,dx + N\,dy$$

$$\text{Flux:}\quad \oint_C \mathbf{F}\cdot\mathbf{n}ds = \oint_C \mathbf{F}\cdot\begin{pmatrix} dy, \\ -dx\end{pmatrix} = \oint_a^b (M,N)\cdot\frac{(dy,-dx)}{dt}\,dt = \int_a^b \left(M\frac{dy}{dt}-N\frac{dx}{dt}\right)dt = \oint_C M\,dy - N\,dx$$

1.2.2 Integrals over surfaces

Let S be a smooth surface in $\mathbb{R}^3$, and let $\mathbf{r}(u,v)$ be a parametrization of S with domain U.

A **weighted area integral** is the integral of a scalar (e.g. a density) over a surface:

$$\iint_S g\,dS = \iint_R g\,|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv$$

A **flux** integral is the flow of a vector field across a surface:

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_U \mathbf{F}\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}|\mathbf{r}_u\times\mathbf{r}_v|\,d\mathbf{u}\,d\mathbf{v} = \iint_U \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,d\mathbf{u}\,d\mathbf{v}$$
$$= \iint_U \big(M(y_u z_v - z_u y_v) + N(z_u x_v - x_u z_v) + P(x_u y_v - y_u x_v)\big)\,d\mathbf{u}\,d\mathbf{v}. \tag{1}$$

(**Remark:** *In the elegant language of differential 2-forms, described in more detail in the appendix, we can write this as:*

$$\iint_U \big(M\,dy\wedge dz + N\,dz\wedge dx + P\,dx\wedge dy\big)\big[\mathbf{r}_u\,du,\ \mathbf{r}_v\,dv\big]$$
$$= \iint_S M\,dy\wedge dz + N\,dz\wedge dx + P\,dx\wedge dy.$$

Here du and dv represent sides of an infinitesimal rectangle in the parameter space U; $\mathbf{r}_u\,du$ and $\mathbf{r}_v\,dv$ are two infinitesimal vectors lying in the surface S; the ordered pair $\big[\mathbf{r}_u\,du,\ \mathbf{r}_v\,dv\big]$ represents the infinitesimal oriented parallelogram they span; and $dx\wedge dy$ operates on this pair by projecting it onto the x-y plane and computing its oriented area, $\det\begin{vmatrix} x_u du & d_v dv \\ y_u du & y_v dv\end{vmatrix} = (x_u y_v - y_u x_v)dudv.$)

1.3 Forms of integrals for manifolds defined by functions

1.4 Types of derivatives

The derivative is an *operator*. An **operator** takes a function as its input and returns a function as its output. Vector calculus works with two kinds of functions: (1) "scalars" (i.e., scalar-valued functions, which map $\mathbb{R}^3$ to $\mathbb{R}$) and (2) "vectors" (i.e., vector-valued functions, which map $\mathbb{R}^3$ to $\mathbb{R}^3$).

Vector calculus has three kinds of (first) derivative. Their precise physical meaning is actually revealed by the appropriate version of the fundamental theorem of calculus. (In fact, these differential operators are defined so as to make their specific version of the fundamental theorem of calculus true).

Name	**symbol**	**input**	**output**	**meaning**
grad	∇	scalar	vector	direction of greatest change and its magnitude
div	$\nabla\cdot$	vector	scalar	tendency of flow to diverge
curl	$\nabla\times$	vector	vector	vorticity of flow (tendency to circulate and direction of circulation)

1.5 Manifestations of the fundamental theorem of calculus

1.5.1 Gradient: work integral of gradient of a potential

$$\int_{\mathbf{r}_a}^{\mathbf{r}_b} \nabla f\cdot d\mathbf{r} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} f_x\,dx + f_y\,dy = \int_{\mathbf{r}_a}^{\mathbf{r}_b} df = f(\mathbf{r}_a) - f(\mathbf{r}_a).$$

1.5.2 Curl: Stokes' theorem

Stokes' theorem states that the flux of the curl through a simply connected surface S is the circulation around the boundary of S:

$$\iint_S dS\, \hat{\mathbf{n}} \cdot \nabla \times \mathbf{F} = \oint_{\partial S} ds \hat{\mathbf{T}} \cdot \mathbf{F};$$

here ∂S denotes the boundary of S; one often omits to write dS (which represents the area of a piece of surface) and ds (which represents the length of a piece of boundary). We note that in a right-handed coordinate system the direction of the circulation is counterclockwise when viewed from the side of the surface toward which $\hat{\mathbf{n}}$ points.

1.5.3 Divergence: Gauss's theorem

Gauss's theorem states that the integral of the divergence over a simply connected region V is the net outward flux through its boundary ∂V:

$$\iiint_V dV\, \nabla \cdot \mathbf{F} = \oiint_{\partial V} dS \hat{\mathbf{n}} \cdot \mathbf{F},$$

where $\hat{\mathbf{n}}$ is the outward-pointing unit normal vector.

A Differential forms

In general to compute an integral expressed with differential forms one must parametrize the curve or surface.

A.1 Differential one-forms

A differential one-forms is used to calculate a work integral. (In two dimensions one-forms are used to calculate the flux across a line, but by rotating the vector field 90 degrees counterclockwise this is equivalent to a work integral.) Consider the generic work integral $\int_C M\, dx + N\, dy$.

Physically dx (i.e. $x_t dt$) represents the x-component of a small displacement $d\mathbf{r}$ (i.e. $\mathbf{r}_t dt$) along an oriented curve C, i.e., the oriented length of the projection of $d\mathbf{r}$ onto the x-axis.

A.2 Differential two-forms

A differential two-form is used to calculate the flux through a surface. Consider the generic flux integral $\iint_S M\, dy \wedge dz + N\, dz \wedge dx + P\, dx \wedge dy$. Physically $dx \wedge dy$ represents the oriented area of the projection of a small piece of surface onto the x-y plane. (Two vectors in the plane span a parallelogram with positive area if the second vector points in a direction that is closer in the counterclockwise direction.)

Consider a surface parametrized by $\mathbf{r}(u, v)$. A small rectangular region of parameter space with dimensions du and dv corresponds to a parallelogram in 3-space with sides $(\mathbf{r}_u du,\ \mathbf{r}_v dv)$. $dx \wedge dy$ operates on this pair of vectors by projecting the parallelogram onto the x-y plane and computing its oriented area:

$$dx \wedge dy(\mathbf{r}_u du, \mathbf{r}_v dv) = \det \begin{vmatrix} x_u du & d_v dv \\ y_u du & y_v dv \end{vmatrix} = (x_u y_v - y_u x_v) du dv.$$

We define the integral

$$I = \iint_S M\,dy \wedge dz + N\,dz \wedge dx + P\,dx \wedge dy$$

by letting it act on a parametrization:

$$\begin{aligned}
I &= \iint_U \big(M\,dy \wedge dz + N\,dz \wedge dx + P\,dx \wedge dy\big)\big[\mathbf{r}_u\,du,\ \mathbf{r}_v\,dv\big] \\
&= \iint_U \big(M(y_u z_v - z_u y_v) + N(z_u x_v - x_u z_v) + P(x_u y_v - y_u x_v)\big)\,d\mathbf{u}\,d\mathbf{v} \\
&= \iint_U \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\,d\mathbf{u}\,d\mathbf{v} \\
&= \iint_U \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}|\mathbf{r}_u \times \mathbf{r}_v|\,d\mathbf{u}\,d\mathbf{v} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,dS.
\end{aligned}$$

This verifies that

$$\iint_S M\,dy \wedge dz + N\,dz \wedge dx + P\,dx \wedge dy = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,dS, \tag{2}$$

where $\mathbf{F} = (M, N, P)$ and $\hat{\mathbf{n}}$ points in the direction determined by the orientation of the parametrization (i.e. the direction of $\mathbf{r}_u \times \mathbf{r}_v$).

But let's look for more physical insight into why this is true. It is enough to show that

$$\iint_U P\,dx \wedge dy = \iint_S P\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,dS,$$

i.e.,

$$\iint_U P\,dx \wedge dy[\mathbf{r}_u du, \mathbf{r}_v dv] = \iint_S P\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,dS, \tag{3}$$

(By symmetry a similar statement will hold for the other components, and adding up the three statements gives (2).)

Consider the oriented parallelpiped with sides $[P\hat{\mathbf{k}},\ \mathbf{r}_u du,\ \mathbf{r}_v dv]$. Its oriented volume is $P\hat{\mathbf{k}} \cdot (r_u du \times r_v dv)$.

But $\hat{\mathbf{n}}\,dS = \mathbf{r}_u du \times \mathbf{r}_v dv$. So the oriented volume is $P\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,dS$; we can interpret this as the volume of the parallelpiped $\Big[P(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}},\ \mathbf{r}_u du,\ \mathbf{r}_v dv\Big]$, where $P\hat{\mathbf{k}}$ has been projected onto $\hat{\mathbf{n}}$, i.e., onto the line perpendicular to the other vectors (such orthogonal projection doesn't affect the volume).

On the other hand, $\Big[P\,dx \wedge dy[\mathbf{r}_u du,\ \mathbf{r}_v dv]\Big]$ is the volume of the parallelpiped $\left[P\hat{\mathbf{k}}, \begin{pmatrix} x_u du \\ y_u du \\ 0 \end{pmatrix}, \begin{pmatrix} x_v dv \\ y_v dv \\ 0 \end{pmatrix}\right]$, where $[\mathbf{r}_u du,\ \mathbf{r}_v dv]$ has been projected onto the plane perpendicular to $P\hat{\mathbf{k}}$ (again, such orthogonal projection doesn't change the volume).

So we conclude that

$$P\,dx \wedge dy[\mathbf{r}_u du, \mathbf{r}_v dv] = P\hat{\mathbf{k}} \cdot (r_u du \times r_v dv) = P\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,dS,$$

as needed to verify (3).