Banach Space Theorems

Thm (Hahn-Banach)

Given:

X vector space.

$p: X \to \mathbb{R}$

p subadditive and homogeneous:

- $p(x+y) \le p(x) + p(y)$
- $p(\alpha x) = \alpha p(x) \quad \forall \alpha \ge 0$.

$S \subseteq X$ subspace

$f: S \to \mathbb{R}$ linear functional

$f(s) \le p(s) \quad \forall s \in S$.

Then

$(\exists F: X \to \mathbb{R}$ linear functional$)$

s.t. $F|_S = f$ and

$\forall x \quad F(x) \le p(x)$

Pf Let $g = f$.

By the Hausdorff Maximal Principle it suffices to show that if $S \subsetneq X$ then g has an extension h to a superspace containing S.

Let $y \in X \setminus S$.

Let $U = \mathrm{span}\{S, y\}$.

Want h an extension of g to U.

For $u \in U$ write $u = \lambda y + s$,

$\lambda \in \mathbb{R}$, $y \in X$, $s \in S$.

[Note: this representation is unique, for if $\lambda_1 y + s_1 = \lambda_2 y + s_2$ then $(\lambda_1 - \lambda_2) y = (s_2 - s_1) \in S$ so $\lambda_1 - \lambda_2 = 0$ and $s_2 - s_1 = 0$.

If h exists, then

$$
\begin{aligned}
h(u) &= h(\lambda y + s) \\
&= \lambda h(y) + h(s) \\
&= \lambda h(y) + g(s).
\end{aligned}
$$

So h is defined by specifying $h(y)$.

Write $\alpha = h(y)$.

Need: $h(\lambda y + s) = \lambda \alpha + g(s) \le p(\lambda y + s)$

$\lambda > 0$: $\alpha \le p\left(y + \frac{s}{\lambda}\right) - g\left(\frac{s}{\lambda}\right) \quad \forall s$

So $\alpha \le p(y+s) - g(s) \quad \forall s$

Case $\lambda < 0$.

Use $\mu p(v) = \mathrm{sgn}(\mu)\, |\mu| p(v)$
$= \mathrm{sgn}(\mu)\, p(|\mu| \cdot v)$

Need $\lambda \alpha \le p(\lambda y + s) - g(s)$

$\alpha \ge (-1) p\left(-y - \frac{s}{\lambda}\right) - g\left(\frac{-s}{\lambda}\right) \quad \forall s$

$\alpha \ge -p(-y - s) - g(s) \quad \forall s$

So need:

$$\sup_s \left[-p(-y-s) - g(s)\right] \le \inf_t \left[p(y+t) - g(t)\right]$$

i.e. $-p(-y-s) - g(s) \le p(y+t) - g(t) \quad \forall s,t$

i.e. $\underbrace{-g(s) + g(t)}_{g(-s+t)} \le \underbrace{p(y+t) + p(-y-s)}_{\ge p(-s+t)}$

Need $g(-s+t) \le p(-s+t)$.

Clear. Done.

Rmk Note that if $\exists s: g(s) = p(s)$, then the choice of α is uniquely determined.

Complex Hahn - Banach

Prop (Rudin 5.17)

V complex vector space.

(a) f complex-linear functional on V

Let $u = \operatorname{Re} f$

Then $f(x) = u(x) - i u(ix)$ $(\forall x \in V)$

(b) Let u be a real-linear functional on V.

Let $f(x) = u(x) - i u(ix)$

Then f is complex-linear functional

(c) Assume V normed.

Suppose $f(x) = u(x) - i u(x)$

Then $\|f\| = \|u\|$.

Pf

Fact Let $z = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$.

Then $\operatorname{Re}(iz) = \operatorname{Re}(i\alpha - \beta) = -\beta$.

So $z = (\operatorname{Re} z) - i \operatorname{Re}(iz)$

Pf (a) Take $z = f(x)$.

So $f(x) = \underbrace{\operatorname{Re}[f(x)]}_{u(x)} - i \underbrace{\operatorname{Re}[\underbrace{if(x)}_{f(ix)}]}_{u(ix)}$

Pf (b) Assume hypotheses.

Clearly $f(x+y) = f(x) + f(y)$,

and $f(rx) = r f(x)$ (for $r \in \mathbb{R}$)

Need $f(ix) = i f(x)$.

LHS $= u(ix) - i u(-x)$

$= u(ix) + i u(x)$

RHS $= i[u(x) - i u(ix)]$

$= i u(x) + u(ix)$ ✓

Pf (c) Assume hyp.

Need $\sup_{\|x\|=1} |f(x)| = \sup_{\|x\|=1} |u(x)|$

Let $\|x\| = 1$. $(\exists \alpha \in \mathbb{C})$ $|f(x)| = \alpha f(x) = f(\alpha x)$

so $= u(\alpha x) \leq \|u\| \cdot \underbrace{\|\alpha x\|}_{=\|x\|}$.

So $\|f\| \leq \|u\|$

But $\|u\| \leq \|f\|$.

So $\|u\| = \|f\|$.

Thm (Rudin 5.16)

X complex normed linear space,

$M \subset X$ subspace,

f complex-linear functional on M.

Then

$(\exists F$ complex linear functional on $M)$

$F|_M = f$ and $\|F\| = \|f\|$.

Pf Assume hypotheses.

Let $u = \operatorname{Re} f$.

By the real Hahn-Banach thm

$(\exists \mathcal{U} : X \to \mathbb{R}$ real linear functional$)$

$\mathcal{U}|_M = u$, and $\|\mathcal{U}\| = \|u\|$

Let $F(x) = \mathcal{U}(x) - i\mathcal{U}(ix)$

By prop 5.17 F is a complex-linear extension of f

and $\|F\| = \|\mathcal{U}\| = \|u\| = \|f\|$.

Done.

Prop

f complex-linear

$u = \operatorname{Re} f$.

Then u is real-linear.

Pf Let $\lambda \in \mathbb{R}$. Need $u(\lambda x) = \lambda(u(x))$

Write $f = u + iv$.

$\lambda f(x) = f(\lambda x)$

so $\lambda u(x) + i\lambda v(x) = u(\lambda x) + i v(\lambda x)$

So $\lambda u(x) = u(\lambda x)$ and $\lambda v(x) = v(\lambda x)$.

Thm (Closed Graph)
X, Y Banach spaces.
A linear map : $X \to Y$,
Graph of A is closed, i.e.:
If $x_n \to x$
and $Ax_n \to y$
Then $y = Ax$

Then A is continuous.
(i.e. if $x_n \to x$
then $Ax_n \to Ax$)

Pf [from Royden]
Let $N(x) = \|x\| + \|Ax\|$.
Claim N is norm and
X is complete in norm N.
(Pf)
positivity ✓
homogeneity ✓
subadditivity ✓
completeness :
Let $N(x_p - x_q) \to 0$.
So $\|x_p - x_q\| \to 0$ and $\|Ax_p - Ax_q\| \to 0$
X and Y are complete.
So $x_n \to x$ and $Ax_n \to y$, some x, y.
Since the graph of A is closed,
$y = Ax$.
So $\|x_n - x\| \to 0$ & $\|A(x_n - x)\| \to 0$.
So $N(x_n - x) \to 0$.
So X is complete in norm N.

Note $\|x\| \leq N(x) \quad \forall x \in X$.
Both norms are complete.
So they are equivalent.
i.e. $(\exists C' > 0)$ $N(x) \leq C' \|x\|$.
i.e. $\|x\| + \|Ax\| \leq C' \|x\|$
So $\|Ax\| \leq C' \|x\|$.
So A is continuous.

Rmk This theorem allows you to move the convergence of Ax_n from the conclusion to the hypothesis when proving the continuity of A.

Thm (Closed Graph)
X, Y Banach spaces

Thm (Open Mapping)
$T \in \mathcal{L}(X,Y)$ (Banach Spaces)
$Y = TX$ (T onto)
Then
T is an open mapping.

Def $B(a,r)$ denotes the open unit ball centered at a of radius r.
$B_X(x,r)$ = ball in X
$B_Y(y,r)$ = ball in Y
$B^r \stackrel{\text{def}}{=} B(0,r)$
$B \stackrel{\text{def}}{=} B(0,1)$

Pf It is necessary and sufficient to show that the image of a ball contains a ball.

Lemma The image of an arbitrary ball contains a ball iff the image of the unit ball contains a ball.

Observe
- $B(x,r) = x + rB$
- $TB(x,r) = Tx + rTB$

Pf of lemma
($\Rightarrow$) $TB_X \supseteq B_Y(y,s) = y + sB_Y$
$\Rightarrow Tx + rTB_X \supseteq Tx + ry + rsB_Y$
$\Rightarrow TB_X(x,r) \supseteq B_Y(Tx+ry, rs)$
($\Leftarrow$) $TB_X(x,r) \supseteq B_Y(y,s)$
$\Rightarrow Tx + rTB_X \supseteq y + sB_Y$
$\Rightarrow TB_X \supseteq \frac{1}{r}(y - Tx + sB_Y)$
$= B_Y(\frac{1}{r}(y-Tx), \frac{s}{r})$

Pf (resumed)
Since T is onto,
$$Y = \bigcup_{k=1}^{\infty} T(kB)$$
So by the Baire category thm, some kB contains an open ball. //

Interesting proposition
The image of the unit ball contains a ball iff it contains a ball centered at 0.

Pf
Suppose $TB_X \supseteq y + sB_Y$
So $TB_X \supseteq -y + sB_Y$
So $\cancel{2}TB_X \supseteq \cancel{2}sB_Y$.
(using Δ inequality).

Thm

Thm $\left(\begin{array}{l}\text{PUB = Principle of Uniform Boundedness}\\ = \text{Banach-Steinhaus Thm}\end{array}\right)$

X Banach space

Y normed space

$\mathcal{F}$ a family of bounded linear operators from X to Y

Then Either:

① $\mathcal{F}$ is uniformly bounded, or

② $\sup_{T\in\mathcal{F}} \|Tx\| = \infty$

$\forall x$ in some dense G_δ in X.

Pf (Rudin $\mathbb{R}$ & $\mathbb{C}$)

Let $\varphi(x) = \sup_{T\in\mathcal{F}} \|Tx\| \quad (\forall x \in X)$.

Note $\varphi(rx) = |r|\varphi(x) \quad \forall x\in X, \forall r\in[0,\infty)$

Let $V_n = \{x : \varphi(x) > n\} \quad (\forall n \in \mathbb{N})$

$= \{x : \exists T \quad \|Tx\| > n\}$

$= \bigcup_{T\in\mathcal{F}} \underbrace{\{x : \|Tx\| > n\}}_{\text{open, since } T \text{ is bdd and } \|\cdot\| \text{ is cont.}}$

So each V_n is open.

Two cases:

① Some V_n is not dense

So V_n^c contains an open ball.

i.e. $\{x : \varphi(x) \le n\} \supset x_0 + rB$

for some $x_0 \in X$, $r \in (0,\infty)$

where B is the unit ball in X.

i.e. $\varphi(x_0 + rB) < n$

Observe:

$\forall x_1, x_2 \in X \quad \forall r \in [0,\infty)$:

- $\varphi(rx_1) = |r|\,\varphi(x_1)$
- $\varphi(x_1 + x_2) = \sup_T \|Tx\| + \sup_T \|Ty\|$

$= \sup_T \|Tx\| + \|Ty\|$

$\ge \sup_T \|T(x+y)\|$

$= \varphi(x_1) + \varphi(x_2)$

(Case 1 cont.)

Thus: $\varphi(x_0 + rB) < n$

$\varphi(-x_0) < n$

$\varphi(rB) < 2n$

So $\varphi(B) < \frac{2n}{r}$

i.e. $\mathcal{F}$ is uniformly bounded by $\frac{2n}{r}$.

② Every V_n is dense.

So $\bigcap_n V_n$ is dense G_δ.

i.e. $\{x : \sup_{T\in\mathcal{F}} \|Tx\| = \infty\}$

is a dense G_δ.

__Thm__

X normed vector space

Y Banach space.

Then $\mathcal{L}(X,Y) = \left\{ \begin{array}{c} \text{bounded linear} \\ \text{operators from} \\ X \text{ to } Y \end{array} \right\}$

is a Banach space.

__Pf__ Write $\mathcal{B} = \mathcal{L}(X,Y)$.

For $B \in \mathcal{B}$

let $\|B\| = \sup_{\|x\|=1} \|Bx\|$.

Then it is easy to verify that:

- Any linear combination of bounded linear operators is a linear operator.
- $\|\cdot\|$ is a norm.

Need $\mathcal{B}$ complete with $\|\cdot\|$.

Let $\{A_n\} \subset \mathcal{B}$ be Cauchy.

So $\|A_n - A_m\| \to 0$.

$\|A_n x - A_m x\| \le \|A_n - A_m\| \cdot \|x\|$,

So $\{A_n x\}$ is Cauchy in Y.

So $(\exists! (Ax))$ $A_n x \to Ax$

Want: $A \in \mathcal{B}$, $A_n \to A$.

A is linear:

- $A(\lambda x) = \lim_{n\to\infty} A_n(\lambda x)$
  $= \lambda \lim_{n\to\infty} A_n x$
  $= \lambda A x$.
- $A(x+y) = \lim_{n\to\infty} A_n(x+y)$
  $= \lim_{n\to\infty} A_n x + \lim_{n\to\infty} A_n y$
  $= Ax + Ay$.

A is bounded:

Since $\{A_n\}$ is Cauchy,

$\exists N \ \forall m,n \ge N \ \|A_n - A_m\| < 1$

So $\forall n > N \ \|A_n\| < \|A_N\| + 1$.

So $\|Ax\| = \lim_{n\to\infty} \|A_n x\|$

$\le (\|A_N\| + 1)\|x\|$,

So A is bounded.

$A_n \to A$:

$\|A_n - A\| = \sup_{\|x\|=1} \|(A_n - A)x\|$

$= \underbrace{\sup_{\|x\|=1} \|A_n x - Ax\|}_{\text{Need } \to 0}$.

well,

$\|A_n x - Ax\| = \lim_{m\to\infty} \|A_n x - A_m x\|$

$\le \overline{\lim_{m\to\infty}} \underbrace{\|A_n - A_m\|}_{\to 0} \cdot \underbrace{\|x\|}_{1}$

So $\|A_n x - Ax\| \to 0$, uniformly in x.

So $\|A_n - A\| \to 0$. Done.