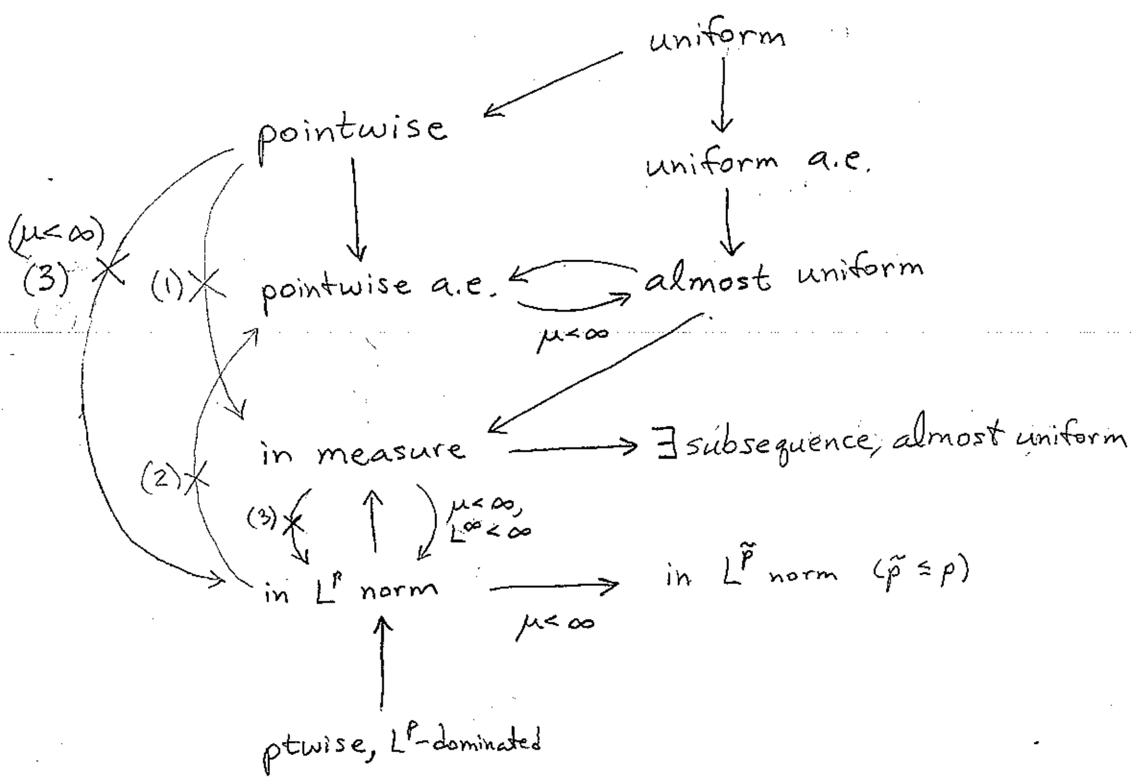
Convergence



Important counterexamples:

① $f_n = \chi_{[n,n+1]} \in L(\mathbb{R})$

$(f_n \to 0$ but $f_n \xrightarrow{m}\!\!\!\!\!/\;\, 0)$

② Let $a_n = \lfloor \log_2 n \rfloor$.

Let $b_n = 2^{a_n}$

Let $c_n = n - b_n$

Let $f_n = \chi_{\left[\frac{c_n}{b_n}, \frac{c_n+1}{b_n}\right]}$

i.e. $f_n = \chi_{\left(\frac{[0,1]+(n-2^{\lfloor \log_2 n\rfloor})}{2^{\lfloor \log_2 n \rfloor}}\right)}$

n	a_n	b_n	c_n	$\chi_{[\ ,\ ]}$
0				
1	0	1	0	$[0,1]$
2	1	2	0	$[0,\frac{1}{2}]$
3	1	2	1	$[\frac{1}{2},1]$
4	2	4	0	$[0,\frac{1}{4}]$
5	2	4	1	$[\frac{1}{4},\frac{1}{2}]$
6	2	4	2	$[\frac{1}{2},\frac{3}{4}]$
7	2	4	3	$[\frac{3}{4},1]$
8	3	8	0	$[0,\frac{1}{8}]$

Note $\int f_n < \frac{2}{n}$,

$f_n : [0,1] \to [0,1]$

(2 cont.)

$f_n \xrightarrow{L^p} 0 \quad (1 \le p < \infty)$

but $f_n \xrightarrow{a.e.}\!\!\!\!\!\!/\;\;\, 0$

③ Let $f_n = n\chi_{(0,\frac{1}{n})}$

So $f_n \to 0$, but

$f_n \xrightarrow{L^p}\!\!\!\!\!/\;\, 0 \quad (1 \le p \le \infty)$

References

- Friedman, Foundations of Modern Analysis

Convergence

Thm (Pointwise, L^p-dominated convergence $\Rightarrow$ convergence in L^p norm)

Pf

Suppose:

- $f_k \to f$ (a.e.)
- $|f_k - f| \le \varphi$ (a.e.)
- $\int \varphi^p < \infty$

Then

$\int |f_k - f|^p \to 0$ by LDCT.

Note

Suppose

- $f_k \to f$ (a.e.)
- $|f_k| \le \varphi$ (a.e.)
- $\int \varphi^p < \infty$

Then

$|f| \le \varphi$ (a.e.)

So $|f_k - f| \le |f_k| + |f| \le 2\varphi$ (a.e.)

Thm (Convergence in L^p norm $\Rightarrow$ convergence in measure)

Pf

Suppose

- $\int |f_k - f|^p \to 0$

Want

- $\forall \varepsilon > 0 \quad \mu\{|f_k - f| > \varepsilon\} \to 0$.

(WLOG $f = 0$.)

$$\underbrace{\int |f_k - f|^p}_{\to 0} > \varepsilon \underbrace{\mu\{|f_k - f|^p > \varepsilon\}}_{\to 0} \quad \forall \varepsilon.$$

Thm ((Convergence in measure $\Rightarrow$ convergence in L^p norm) for a uniformly bounded sequence of functions on a set of finite measure.)

Pf Let $1 \le p < \infty$

Let $|f_k| < M \quad \forall k$.

Let $\int = \int_E$, $\mu(E) < \infty$.

Lemma 1 (Let $p > 0$. $f_k \to 0$ in measure iff $|f_k|^p \to 0$ in measure.)

Pf

$f_k \to 0$ in measure

$\Leftrightarrow \forall \varepsilon > 0 \quad \mu\{|f_k| > \varepsilon\} \to 0$

$\Leftrightarrow \forall \varepsilon^p > 0 \quad \mu\{|f_k|^p > \varepsilon^p\} \to 0$

$\Leftrightarrow \forall \varepsilon > 0 \quad \mu\{|f_k|^p > \varepsilon\} \to 0$,

Lemma 2 ((Convergence in measure $\Rightarrow$ convergence in mean) for a uniformly bounded sequence of functions on a set of finite measure.)

Pf

Suppose $\forall \varepsilon > 0 \ \exists K \ \forall k > K$

$\mu\{|f_k| > \varepsilon\} < \varepsilon$.

Then $\int |f_k| < M \cdot \varepsilon + \varepsilon \mu(E)$

$= \varepsilon(M + \mu(E))$

$\to 0$ as $\varepsilon \to 0$

Pf of thm

Suppose $f_k \to f$ in measure.

So $|f_k - f| \to 0$ in measure.

So $|f_k - f|^p \to 0$ in measure by Lemma 1

So $\int |f_k - f|^p \to 0$ by lemma 2.

Thm Let $1 \le \tilde{p} \le p$.
Let $\int 1 < \infty$. (finite measure)

Then $\boxed{\text{convergence in } L^p \text{ norm} \Rightarrow \text{convergence in } L^{\tilde{p}} \text{ norm}}$

Pf

Let $\int |f_k - f|^p \to 0$. Let $\tilde{f}_k = f_k - f$.

Write $|\tilde{f}_k| = g_k + h_k$, $0 \le g_k \le 1$, $h_k \ge 1$ or 0.

Then $\|\tilde{f}_k\|_{\tilde{p}} \le \|g_k\|_{\tilde{p}} + \|h_k\|_{\tilde{p}}$.

But $|h_k|^{\tilde{p}} \le |h_k|^p \le |\tilde{f}_k|^p$.

So $\int |h_k|^{\tilde{p}} \to 0$.

Since $|g_k|^p \le |\tilde{f}_k|^p$, $\|g_k\|_p \to 0$.

So $g_k \to 0$ in measure. Since (g_k) is uniformly bounded by 1 on a set of finite measure, $\|g_k\|_{\tilde{p}} \to 0$.

So $\|\tilde{f}_k\|_{\tilde{p}} \to 0$, as needed.

COR

Let $1 \le \tilde{p} \le p$

Let $\int 1 < \infty$ (finite measure)

Then $\exists C \ \forall f$

$$\|f\|_{\tilde{p}} \le C \|f\|_p .$$

Pf

Suppose false.

Let $\|f_k\|_{\tilde{p}} > 4^k \|f_k\|_p$,

$\|f_k\|_p = \frac{1}{2^k}$.

Then $f_k \to 0$ in L^p norm

but f_k is not convergent in $L^{\tilde{p}}$ norm

Contradicting the previous thm.

Thm

Let $\mu(E) < \infty$

Let $1 \le \tilde{p} \le p$

Then

$\|f\|_{\tilde{p}} \le C \|f\|_p$

where $C =$

Convergence

Kinds of convergence

Let f and (f_k) be functions from a set S to a metric space.

① $f_k \to f$ <u>pointwise</u> iff

$$\forall x \in S \ \forall \varepsilon > 0 \ \exists K \ \forall k > K \quad d(f_k(x), f(x)) < \varepsilon$$

② $f_k \xrightarrow{u} f$ (<u>uniformly</u>) iff

$$\forall \varepsilon > 0 \ \exists K \ \forall k > K \ \forall x \in S \quad d(f_k(x), f(x)) < \varepsilon$$

Let f and $(f_k)_{k \in \mathbb{N}}$ be measurable functions from a measure space (Ω, Σ, μ) to (a metric space?).

③ $f_k \xrightarrow{m} f$ (<u>in measure</u>)

$$\Leftrightarrow \forall \varepsilon > 0 \ \lim_{k \to \infty} \mu\{|f_k - f| > \varepsilon\} = 0$$

$$\Leftrightarrow \forall \varepsilon > 0 \ \forall \eta > 0 \ \exists K \ \forall k > K \quad \mu\{|f_k - f| > \varepsilon\} < \eta$$

$$\Leftrightarrow \forall \varepsilon > 0 \ \exists K \ \forall k > K \quad \mu\{|f_k - f| > \varepsilon\} < \varepsilon$$

Let N be a norm (or metric?) on a (complete?) space $\mathcal{B}$ of measurable functions.

Let f and $(f_k)_{k \in \mathbb{N}} \in \mathcal{B}$.

④ $f_k \to f$ <u>in the norm</u> N

$\Leftrightarrow N(f_k - f) \to 0$ as $k \to \infty$.

Example of normed space:

$L^p(\Omega)$, $1 \le p \le \infty$.

$$\|f\|_p = \begin{cases} \left(\int_\Omega |f|^p\right)^{\frac{1}{p}} & \text{if } p < \infty \\ \sup\{M;\ \mu\{|f| > M\} > 0\} & \text{if } p = \infty \\ \text{(the essential supremum of } f\text{)}. \end{cases}$$

⑤ $f_k \xrightarrow{a.u.} f$ (<u>almost uniformly</u>)

$\Leftrightarrow f_k \xrightarrow{u} f$ except on a set of arbitrarily small measure

[Old]

Relationships among kinds of convergence.

- Uniform convergence a.e. = convergence in L^∞ norm.
- uniform convergence $\Rightarrow$ almost uniform convergence
- almost uniform convergence $\Rightarrow$ pointwise convergence.

(Wheeden-Zygmund pp 56-61)

* <u>If $|\Omega| < \infty$</u>:

- pointwise convergence a.e. $\Leftrightarrow$ almost uniform convergence (Egorov) ($\Rightarrow$ convergence in measure)

- Almost uniform convergence $\Rightarrow$ convergence in measure
- Pointwise convergence $\not\Rightarrow$ convergence in measure

- Cauchy in measure $\Rightarrow \exists$ an almost uniformly convergent subsequence

- Cauchy criterion for convergence in measure: f_k converges in measure on Ω iff $\forall \varepsilon > 0 \ \lim_{k,l \to \infty} \mu\{|f_k - f_l| > \varepsilon\} = 0$

Convergence involving integral norms

* If $|f_k - f|^p < \varphi^p$, $\int \varphi^p < \infty$, then LDCT says: a.e. pointwise convergence of a dominated sequence of functions $\Rightarrow$ convergence in the L^p norm ($p < \infty$)

(• convergence in L^p norm ($1 \le p \le \infty$) $\Rightarrow \exists$ a.e. pointwise convergent subsequence (by proof of completeness of L^p).)

* <u>If $\mu(\Omega) < \infty$</u> and $p_1 < p_2$

- convergence in L^{p_2} $\Rightarrow$ convergence in L^{p_1} (Write $(f_k - f) = g_k + h_k$, $g_k \le 1$, $h_k \ge 1$ or 0.)

- convergence in L^p norm ($1 \le p \le \infty$) $\Rightarrow$ convergence in measure.

Generalized Norm-Builder

Let N_α be a uniformly equivalent family of semi-norms indexed by A, one of which is actually a norm

i.e. Suppose $\exists C_2 \ \forall \alpha \in A$

i.e. $\frac{1}{C_2} N_\alpha(x) \le \tilde{N}_\alpha(x) \le C_2 N_\alpha(x)$

Let $M, \tilde{M}$ be norms on $\mathbb{R}^A$ s.t. $\exists C_1 \ \forall y \in \mathbb{R}^A$

$$\frac{1}{C_1} M(y) \le \tilde{M}(y) \le C_1 M(y)$$

and,

~~Let $L(x \mapsto (\alpha \mapsto$~~

Let $L(x) = M(\alpha \mapsto N_\alpha(x))$,

$\tilde{L}(x) = \tilde{M}(\alpha \mapsto \tilde{N}_\alpha(x))$

claim L and $\tilde{L}$ are compatible norms

M norm means:

$M(y_1+y_2) \le M(y_1) + M(y_2)$

i.e. $M((\alpha \mapsto y_1(\alpha)) + (\alpha \mapsto y_2(\alpha)))$

$\le M(\alpha \mapsto y_1(\alpha)) + M(\alpha \mapsto y_2(\alpha))$

Norms:

$L(sx) = M(\alpha \mapsto N_\alpha(sx))$

$= M(\alpha \mapsto |s| N_\alpha(x))$

$= |s| M(\alpha \mapsto N_\alpha(x))$

$L(x+y) = M(\alpha \mapsto N_\alpha(x+y))$

$= M(\alpha \mapsto (\text{something} \le N_\alpha(x) + N_\alpha(y)))$

$\overset{?}{\le} M(\alpha \mapsto N_\alpha(x)) + M(\alpha \mapsto N_\alpha(y))$ by ~~lemma~~ additional hypothesis that M is increasing in each index

$= L(x) + L(y)$

Hypothesis ~~Lemma~~

Suppose $0 < f_\alpha < g_\alpha \ \forall \alpha$. i.e. $g_\alpha = f_\alpha + \tilde{f}_\alpha$, all pos.

Then $M(f) \le M(g)$

positivity: $L(x) = M(\alpha \mapsto N_\alpha(x)) > 0$ by positivity of M and positivity of some N_α.

equivalence.

$\tilde{L}(x) = \tilde{M}(\alpha \mapsto \tilde{N}_\alpha(x)) = \tilde{M}(\alpha \mapsto \text{something} \le C_2 N_\alpha(x)))$

$= \tilde{M}(\text{something} \le \alpha \mapsto C_2 N_\alpha(x))$

$= \tilde{M}(\text{something} \le C_2 (\alpha \mapsto N_\alpha(x)))$

$\le C_2 \tilde{M}(\alpha \mapsto N_\alpha(x)) \le C_2 C_1 M(\alpha \mapsto N_\alpha(x))$

aa3

$$\int \prod_i a_i \le \prod_i \left(\int |a_i|^{p_i} \right)^{\frac{1}{p_i}}$$

$$\int \prod_i |a_i|^{\frac{1}{p_i}} \le \prod_i \left(\int |a_i| \right)^{\frac{1}{p}}$$

$$\|u\|_{W^{k,p}(U)} = \left(\sum_{|\alpha| \le k} \int_U |D^\alpha u|^p \, dx \right)^{\frac{1}{p}}$$

$$\|Du\|_p = \| |Du| \|_p$$

$$= \left\| \left(\sum_{|\alpha|=1} |D^\alpha u|^2 \right)^{\frac{1}{2}} \right\|$$

$$= \left(\int \left(\sum_{|\alpha|=1} |D^\alpha u|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}$$

$$= \left(\int \left(C \left(\sum |D^\alpha u|^p \right)^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}}$$

$$= \left(\int C^p \sum |D^\alpha u|^p \right)^{\frac{1}{p}}$$

$$= C \left(\int \sum |D^\alpha u|^p \right)^{\frac{1}{p}}$$

equivalent to sobolev norm

$$|Du|_{\#} = |Du|_2$$

$$\approx C |Du|_p$$

$q < p$, $\mu(E) < \infty$

$\|\varepsilon\|_q = \left(\int \varepsilon^q\right)^{\frac{1}{q}}$

$= \left(\mu(E)\,\varepsilon^q\right)^{\frac{1}{q}}$

$= \mu(E)^{\frac{1}{q}}\cdot\varepsilon$

Want $\|\varepsilon\|_q \le C\|\varepsilon\|_p$

i.e. $\mu(E)^{\frac{1}{q}}\cancel{\varepsilon} \le C\,\mu(E)^{\frac{1}{p}}\cancel{\varepsilon}$

$C \ge \mu(E)^{\frac{1}{q}-\frac{1}{p}}$

$\frac{1}{q} > \frac{1}{p}$

$\frac{1}{q} - \frac{1}{p} > 0$

$\varepsilon \le f \le r\varepsilon$ Say $r = 2$

$\|f\|_q = \left(\int f^q\right)^{\frac{1}{q}}$

$\in \left(\int [\varepsilon, r\varepsilon]^q\right)^{\frac{1}{q}}$

$\in \left(\int [\varepsilon^q, r^q\varepsilon^q]\right)^{\frac{1}{q}}$

$\in \left(\varepsilon^q[1, r^q]\,\mu(E)\right)^{\frac{1}{q}}$

$\in \varepsilon\cdot[1, r]\,\mu(E)^{\frac{1}{q}}$

$f = g + h$

$\|g\|_{\tilde{p}} \le C\|g\|_p$

$\|h\|_{\tilde{p}} \le C\|h\|_p$

$\|f\|_{\tilde{p}} = \|g+h\|_{\tilde{p}}$

$\le \|g\|_{\tilde{p}} + \|h\|_{\tilde{p}}$

$\le C(\|g\|_p + \|h\|_p)$

$\int \|f\|_q = \left(\int |f|^q\right)^{\frac{1}{q}}$

$\in \left(\sum_n \underbrace{\mu\{r^n \le |f| < r^{n+1}\}}_{\mu_n}\left(r^n[1,r]\right)^q\right)^{\frac{1}{q}}$

$\le \left(\sum_n \mu(E)\left(r^n[1,r]\right)^q\right)^{\frac{1}{q}}$

$= \mu(E)\,[1, r]\left(\left(\sum_n r^{nq}\right)^{\frac{1}{q}}\right)$

$\sum_n a^n = \frac{1}{1-a}$ if $|a| < 1$